Operator Theory: Advances and
Applications
Vol. 165

Interpolation, Schur Functions and Moment Problems

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Editors

Linear
Operators &
Linear
Systems

Birkhäuser Verlag
Basel · Boston · Berlin

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2000 Mathematics Subject Classification 34A55, 47A57, 47B35

A CIP catalogue record for this book is available from the
Library of Congress, Washington D.C., USA

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

ISBN 3-7643-7546-9 Birkhäuser Verlag, Basel – Boston – Berlin

Part of Springer Science+Business Media
Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Cover design: Heinz Hiltbrunner, Basel
Printed in Germany
ISBN-10: 3-7643-7546-9
ISBN-13: 978-3-7643-7546-1
e-ISBN: 3-7643-7547-7

9 8 7 6 5 4 3 2 1

www.birkhauser.ch

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Editorial Introduction

The present volume, entitled "Interpolation, Schur functions and moment problems", is the second in the new subseries LOLS (*Linear Operators and Linear Systems* of the series *Operator Theory: Advances and Applications*). The main part of this volume is a selection of essays on various aspects of what is by some authors called *Schur analysis*.

To present the papers and set the volume into perspective, let us recall that a function analytic and contractive in the open unit disk is called a Schur function. In 1917, Schur associated to such a function a sequence, finite or infinite, of numbers in the open unit disk $\mathbb{D}$, called Schur coefficients. One can associate such a sequence also to a function analytic and with a positive real part in $\mathbb{D}$. Such functions are called Carathéodory functions and the associated coefficients are sometimes called Verblunsky coefficients. Carathéodory functions appear in the trigonometric moment problems via the Herglotz representation formula. Carathéodory and Schur functions have no poles in the open unit disk. Allowing functions with poles in $\mathbb{D}$ was first considered by Takagi in his 1924 paper [7]. Functions of the form $s(z) = \frac{p(z)}{z^n p(1/z^*)^*}$ (where $p(z)$ is a polynomial of degree n) play an important role in that paper, and are a particular instance of what was later known as generalized Schur functions. These are functions meromorphic in $\mathbb{D}$ and such that the kernel $\frac{1-s(z)s(w)^*}{1-zw^*}$ has a finite number of negative squares in the domain of holomorphy of s. Generalized Schur functions have been introduced independently (and in different ways) by M.G. Kreĭn and H. Langer [5] (these authors also defined in a similar way generalized Carathéodory functions) and by C. Chamfy and Dufresnoy [3], [2]. The theory of Schur and generalized Schur functions also make sense in the matrix and operator-valued cases, and are a continuous source of new problems, as is illustrated in the papers presented in this volume. We note that the translation of the papers of Schur and research papers on the Schur algorithm form the contents of volume 16 of the series OTAA, see [4] and that operator-valued generalized Schur functions have been studied in the volume 96 of the series OTAA, see [1].

Now we can say that under the word Schur analysis one encounters the variety of problems related to Schur and Carathéodory functions such as interpolation problems, moment problems, study of the relationships between the Schur coefficients and the properties of the function, study of underlying operators,... Such questions are also considered in the setting of generalized Schur and generalized Carathéodory functions, and in the "line case", where functions analytic in a halfplane rather than in the open unit disk are considered and where Hankel operators replace Toeplitz operators.

The volume contains seven papers, and we now review their contents:

Boundary interpolation of generalized Schur functions: In the paper "*Basic boundary interpolation for generalized Schur functions and factorization of rational J-unitary matrix functions*" by **D. Alpay, A. Dijksma, H. Langer** and **G. Wanjala**, the authors develop the counterpart of the Schur algorithm for a generalized Schur function at a boundary point. This approach allows to solve the so-called *basic interpolation problem* introduced in earlier work for an inner point. In the paper "*Boundary Nevanlinna–Pick interpolation problems for generalized Schur functions*", **V. Bolotnikov** and **A. Kheifets** solve three different multipoints boundary interpolation problems. In both papers the problems take into account the particularities of the nonpositive case and have no direct analog in the positive case.

Discrete first-order systems: In a previous paper (which appeared in the first volume of the LOLS subseries), **D. Alpay** and **I. Gohberg** introduced the characteristic spectral functions associated to a discrete first order systems. The paper "*Discrete analogs of canonical systems with pseudo-exponential potential. Inverse problems*" continues this study and focuses on inverse problems. An important role is played by the solutions of an underlying Nehari interpolation problem which take unitary values on the unit circle and which admit a generalized Wiener–Hopf factorization.

Schur parameters of pseudocontinuable Schur functions: In the paper "*Shift operators contained in contractions, Schur parameters and pseudocontinuable Schur functions*", **V.K. Dubovoy** studies relationships between the maximal shift and coshift operator of a completely non unitary contraction. A main result in the paper is the characterisation of sequence of Schur coefficients for Schur functions which are not inner but admit a pseudo-analytic continuation of bounded type in the exterior of the open unit disk. The methods of the paper are an illustration of the feedback between function theory and operator theory methods.

The matrix-valued case: The matrix-valued case has difficulties of its own, in particular in the degenerate cases. In the paper "*A Truncated Matricial Moment Problem on a Finite Interval*", **A. Choque Rivero, Y. Dyukarev, B. Fritzsche** and **B. Kirstein** use Potapov's method of the Fundamental Matrix Inequality (FMI) to solve a matrix truncated moment problem on an interval. The scalar case had been considered by M.G. Kreĭn and A. Nudelman (see [6]). A complete description of the set of solutions is given in the strictly positive case. In the paper "*The Matricial Carathéodory Problem in Both Nondegenerate and Degenerate Cases*", **B. Fritzsche, B. Kirstein** and **A. Lasarow** develop a new approach to the matricial Carathéodory interpolation problem.

Inversion formula: In the paper "*A Gohberg-Heinig type inversion formula involving Hankel operators*", **G.J. Groenewald** and **M.A. Kaashoek** prove a formula for the inverse of an operator of the form $I - K_1K_2$ where K_1 and K_2 are Hankel operators between matricial L_1 spaces. The proof is given first for kernel functions of stable exponential type, and then uses an approximation argument. In the first step the state space method is used.

We note that the fourth and seventh papers are related to the line case, while the others deal with the disk case. This ends a short review of this volume.

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