ISNM
International Series of Numerical Mathematics
Volume 153

Nonlinear Partial Differential Equations with Applications

Tomáš Roubíček

Birkhäuser Verlag
Basel · Boston · Berlin

Author:

Tomáš Roubíček
School of Mathematics
Charles University
Faculty of Mathematics and Physics
Sokolovská 83
186 75 Praha 8 - Karlin
Czech Republic

2000 Mathematics Subject Classification: Primary 35Jxx, 35Kxx, 35Qxx, 47Hxx, 47Jxx, 49Jxx; Secondary 65Mxx, 65Nxx, 74Bxx, 74Fxx, 76Dxx, 76Rxx, 80Axx

A CIP catalogue record for this book is available from the Library of Congress, Washington D.C., USA
Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

ISBN 3-7643-7293-1 Birkhäuser Verlag, Basel – Boston – Berlin

Part of Springer Science+Business Media
Printed on acid-free paper produced from chlorine-free pulp. TCF∞
Printed in Germany

ISBN-10: 3-7643-7293-1
ISBN-13: 978-3-7643-7293-4
e-ISBN: 3-7643-7397-0

9 8 7 6 5 4 3 2 1
www.birkhauser.ch

To the memory of professor Jindřich Nečas

Contents

II EVOLUTION PROBLEMS 185

Preface

The theoretical foundations of differential equations have been significantly developed, especially during the 20th century. This growth can be attributed to fast and successful development of supporting mathematical disciplines (such as functional analysis, measure theory, and function spaces) as well as to an ever-growing call for applications especially in engineering, science, and medicine, and ever better possibility to solve more and more complicated problems on computers due to constantly growing hardware efficiency as well as development of more efficient numerical algorithms.

A great number of applications involve distributed-parameter systems (which can be, in particular, described by *partial differential equations*[1]) often involving various nonlinearities. This book focuses on the theory of such equations with the aim of bringing it as fast as possible to a stage applicable to real-world tasks. This competition between rigor and applicability naturally needs many compromises to keep the scope reasonable. As a result (or, conversely, the reason for it) the book is primarily meant for graduate or PhD students in programs such as mathematical modelling or applied mathematics. Although some preliminary knowledge of modern methods in linear partial differential equations is useful, the book is basically self-contained if the reader consults Chapter 1 where auxiliary material is briefly presented without proofs.

The prototype tasks addressed in this book are boundary-value problems for *semilinear*[2] equations of the type

$$-\Delta u + c(u) = g, \qquad \text{or more general} \qquad -\operatorname{div}\big(\kappa(u)\nabla u\big) + c(u) = g, \tag{0.1}$$

or, still more general, for *quasilinear*[3] equations of the type

$$-\operatorname{div}\big(a(u,\nabla u)\big) + c(u,\nabla u) = g, \tag{0.2}$$

[1]The adjective "partial" refers to occurrence of partial derivatives.

[2]In this book the adjective "semilinear" will refer to equations where the highest derivatives stand linearly and the induced mappings on function spaces are weakly continuous.

[3]The adjective "quasilinear" refers to equations where the highest derivatives occur linearly but multiplied by functions containing lower-order derivatives, which means here the form $-\sum_{i,j=1}^{n} a_{ij}(x,u,\nabla u)\partial^2 u/\partial x_i \partial x_j + c(x,u,\nabla u) = g$. After applying the chain rule, one can see that (0.2) is only a special case, namely an equation in the so-called divergence form.

and various generalizations of those equations, in particular variational inequalities. Furthermore, systems of such equations are treated with emphasis on various real-world applications in (thermo)mechanics of solids and fluids, in electrical devices, engineering, chemistry, biology, etc. These applications are contained in Part I.

Part II addresses evolution variants of previously treated boundary-value problems like, in case of (0.2),[4]

$$\frac{\partial u}{\partial t} - \mathrm{div}\big(a(u,\nabla u)\big) + c(u,\nabla u) = g, \tag{0.3}$$

completed naturally by boundary conditions and initial or periodic conditions.

Let us emphasize that our restriction on the quasilinear equations (or inequalities) in the divergence form is not severe from the viewpoint of applications. However, in addition to fully nonlinear equations of the type $a(\Delta u) = g$ or $\frac{\partial}{\partial t}u + a(\Delta u) = g$, topics like problems on unbounded domains, homogenization, detailed qualitative aspects (asymptotic behaviour, attractors, blow-up, multiplicity of solutions, bifurcations, etc.) and, except for a few remarks, hyperbolic equations are omitted.

In particular cases, we aim primarily at formulation of a suitable definition of a solution and methods to prove existence, uniqueness, stability or regularity of the solution.[5] Hence, the book balances the presentation of general methods and concrete problems. This dichotomy results in two levels of discourse interacting with each other throughout the book:

- abstract approach – can be explained systematically and lucidly, has its own interest and beauty, but has only an auxiliary (and not always optimal) character from the viewpoint of partial differential equations themselves,
- targeted concrete partial differential equations – usually requires many technicalities, finely fitted with particular situations and often not lucid.

The addressed methods of general purpose can be sorted as follows:

- ◦ indirect in a broader sense: construction of auxiliary approximate problems easier to solve (e.g. Rothe method, Galerkin method, penalization, regularization), then a-priori estimates and a limit passage;
- ◦ direct in a broader sense: reformulation of the differential equation or inequality into a problem solvable directly by usage of abstract theoretical results, e.g. potential problems, minimization by compactness arguments;

[4] In fact, a nonlinear term of the type $c(u)\frac{\partial}{\partial t}u$ can easily be considered in (0.3) instead of $\frac{\partial}{\partial t}u$; see p. 253 for a transformation to (0.3) or Sect. 11.2 for a direct treatment. Besides, nonlinearity like $C(\frac{\partial}{\partial t}u)$ will be considered, too; cf. Sect. 11.1.1 or 11.1.2.

[5] To complete the usual mathematical-modelling procedure, this scheme should be preceded by a formulation of the model, and followed by numerical approximations, numerical analysis, with computer implementation and graphic visualization. Such, much broader ambitions are not addressed in this book, however.

◦ iterational: fixed points, e.g. Banach or Schauder's theorems;

We make the general observation that simple problems usually allow several approaches while more difficult problems require sophisticated combination of many methods, and some problems remain even unsolved.

The material in this book is organized in such a way that some material can be skipped without losing consistency. At this point, Table 1 can give a hint:

	steady-state	evolution
basic minimal scenario	Chapters 2,4	Chapter 7, Sect. 8.1–8.8
variational inequalities	Chapter 5	Chapter 10
accretive setting	Chapter 3	Chapter 9
systems of equations	Chapters 6	Chapter 12, Sect. 9.5
some special topics	—	Sect. 8.9–8.10, Chapter 11
auxiliary summary of general tools	Chapter 1	

Table 1. *General organization of this book.*

Except for the basic minimal scenario, the rest can be combined (or omitted) quite arbitrarily, assuming that the evolution topics will be accompanied by the corresponding steady-state part. Most chapters are equipped with exercises whose solution is mostly sketched in footnotes. Suggestions for further reading as well as some historical comments are in biographical notes at the ends of the chapters.

The book reflects both my experience with graduate classes I taught in the program "Mathematical modelling" at Charles University in Prague during 1996–2005[6] and my own research[7] and computational activity in this area during the past (nearly three) decades, as well as my electrical-engineering background and research contacts with physicists and material scientists. My thanks and deep

[6]In the usual European 2-term organization of an academic year, a natural schedule was Part I (steady-state problems) for one term and Part II (evolution problems) for the other term. Yet, only a selection of about 60% of the material was possible to expose (and partly accompanied by exercises) during a 3-hour load per week for graduate- or PhD-level students. Occasionally, I also organized one-term special "accretive-method" course based on Chapters 3 and 9 only.

[7]It concerns in particular a research under the grants 201/03/0934 (GA ČR), IAA 1075402 (GA AV ČR), and MSM 21620839 (MŠMT ČR) whose support is acknowledged.

gratitude are to a lot of my colleagues, collaborators, or tutors, in particular M. Arndt, M. Beneš, M. Bulíček, M. Feistauer, J. Franců, J. Haslinger, K.-H. Hoffmann, J. Jarušek, J. Kačur, M. Kružík, J. Málek, J. Malý, A. Mielke, J. Nečas, M. Pokorný, D. Pražák, A. Świerczewska, and J. Zeman for numerous discussions or/and reading the manuscript.

Praha, 2005 T.R.

Notational conventions

A	a mapping (=a nonlinear operator), usually $V \to V^*$ or $\mathrm{dom}(A) \to X$,
a.a., a.e.	a.a.=almost all, a.e.=almost each, referring to Lebesgue measure,
$C(\bar{\Omega})$	the space of continuous functions on $\bar{\Omega}$, equipped with the norm $\|u\|_{C(\bar{\Omega})} = \max_{x\in\bar{\Omega}} \|u(x)\|$, sometimes also denoted by $C^0(\bar{\Omega})$,
$C^{0,1}(\Omega)$	the space of the Lipschitz continuous functions on Ω,
$C(\bar{\Omega};\mathbb{R}^n)$	the space of the continuous $\mathbb{R}^n$-valued functions on $\bar{\Omega}$,
$C^k(\bar{\Omega})$	the space of functions whose all derivatives up to k-th order are continuous on $\bar{\Omega}$,
$\mathrm{cl}(\cdot)$	the closure,
$\mathcal{D}(\Omega)$	the space of infinitely smooth functions with a compact support in Ω, see p. 10,
$\mathrm{D}\Phi(u,v)$	the directional derivative of Φ at u in the direction v,
$\mathrm{diam}(S)$	the diameter of a set $S \subset \mathbb{R}^n$; i.e. $\mathrm{diam}(S) := \sup_{x,y\in S} \|x-y\|$,
div	the divergence of a vector field; i.e. $\mathrm{div}(v) = \frac{\partial}{\partial x_1}v_1 + \cdots + \frac{\partial}{\partial x_n}v_n$ for $v=(v_1,\ldots,v_n)$,
$\mathrm{dom}(A)$	the definition domain of the mapping A; in case of a set-valued mapping $A: V_1 \rightrightarrows V_2$ we put $\mathrm{dom}(A) := \{v\in V_1;\ A(v) \neq \emptyset\}$,
$\mathrm{dom}(\Phi)$	the domain of $\Phi: V \to \mathbb{R}\cup\{+\infty\}$; $\mathrm{dom}(\Phi) := \{v\in V;\ \Phi(v) < +\infty\}$,
$\mathrm{epi}(\Phi)$	the epigraph of Φ; i.e. $\{(v,a)\in V\times\mathbb{R};\ a \geq \Phi(v)\}$,
I	the time interval $[0,T]$,
$\mathbf{I}$	the identity mapping,
$\mathbb{I}$	the unit matrix,
$\mathrm{int}(\cdot)$	the interior,
J	the duality mapping,
$\mathcal{L}(V_1,V_2)$	the Banach space of linear continuous mappings $A: V_1 \to V_2$ normed by $\|A\|_{\mathcal{L}(V_1,V_2)} = \sup_{\|v\|_{V_1}\leq 1} \|Av\|_{V_2}$,
$L^p(\Omega)$	the Lebesgue space of p-integrable functions on Ω, equipped with the norm $\|u\|_{L^p(\Omega)} = \big(\int_\Omega \|u(x)\|^p \mathrm{d}x\big)^{1/p}$,
$L^p(\Omega;\mathbb{R}^n)$	the Lebesgue space of $\mathbb{R}^n$-valued p-integrable functions on Ω,
$\mathcal{M}(\bar{\Omega})$	the space of regular Borel measures, $\mathcal{M}(\bar{\Omega}) \cong C(\bar{\Omega})^*$, cf. p.10,
$\mathrm{meas}_n(\cdot)$	n-dimensional Lebesgue measure of a set,
n	the spatial dimension,
$\mathbb{N}$	the set of all natural numbers,
$\mathcal{N}_a$	the Nemytskiĭ mapping induced by an integrand a,

$N_K(\cdot)$	the normal cone, c.f. 6,
$\mathcal{O}(\cdot)$	the "great O" symbol: $f(\varepsilon)=\mathcal{O}(\varepsilon^\alpha)$ for $\varepsilon\searrow 0$ means $\limsup_{\varepsilon\searrow 0}\dfrac{\lvert f(\varepsilon)\rvert}{\varepsilon^\alpha}<\infty$,
$o(\cdot)$	the "small O" symbol: $f(\varepsilon)=o(\varepsilon^\alpha)$ for $\varepsilon\searrow 0$ means $\lim_{\varepsilon\searrow 0} f(\varepsilon)/\varepsilon^\alpha=0$,
p	the exponent related to the polynomial growth/coercivity of the highest-order term in a differential operator,
$p'=\frac{p}{p-1}$	the conjugate exponent to $p\in[1,+\infty]$, cf. (1.20) on p.12,
p^*	the exponent in the embedding $W^{1,p}(\Omega)\subset L^{p^*}(\Omega)$, see (1.34)on p.16,
p^{**}	the exponent in the embedding $W^{2,p}(\Omega)\subset L^{p^{**}}(\Omega)$, i.e. $p^{**}=(p^*)^*$,
$p^\#$	the exponent in the trace operator $u\mapsto u\lvert_\Gamma : W^{1,p}(\Omega)\to L^{p^\#}(\Gamma)$, see (1.37); e.g. $p^{\#\prime}$ or $p^{*\#\prime}$ mean $(p^\#)'$ or $((p^*)^\#)'$, respectively,
$p^{\circledast}$	the exponent in the continuous embedding $L^p(I;W^{1,p}(\Omega))\cap L^\infty(I;L^2(\Omega))\subset L^{p^{\circledast}}(Q)$, see (8.116) on p.233,
Q	a space-and-time cylinder, $Q=I\times\Omega$,
$\mathbb{R}, \mathbb{R}^+, \mathbb{R}^-$	the set of all (resp. positive, or negative) reals,
$\mathbb{R}^n$	the Euclidean space with the norm $\lvert s\rvert=\lvert(s_1,\ldots,s_n)\rvert=\big(\sum_{i=1}^n s_i^2\big)^{1/2}$,
sign	the single-valued "signum", i.e. the mapping $\mathbb{R}\to[-1,1]$, $\mathrm{sign}(0)=0$, $\mathrm{sign}(\mathbb{R}^+)=1$, $\mathrm{sign}(\mathbb{R}^-)=-1$, cf. Figure 10a on p.125,
Sign	the set-valued "signum", i.e. the mapping $\mathbb{R}\rightrightarrows[-1,1]$, $\mathrm{Sign}(0)=[-1,1]$, $\mathrm{Sign}(\mathbb{R}^+)=\{1\}$, $\mathrm{Sign}(\mathbb{R}^-)=\{-1\}$, cf. Figure 10b on p.125,
span$(\cdot)$	the linear hull of the specified set,
supp(u)	the support of a function u, i.e. the closure of $\{x\in\Omega;\ u(x)\neq 0\}$,
T	a fixed time horizon, $T>0$,
V	a separable reflexive Banach space (if not said otherwise), $\lVert\cdot\rVert_V$ (or briefly $\lVert\cdot\rVert$) its norm,
V^*	a topological dual space with $\lVert\cdot\rVert_{V^*}$ (or briefly $\lVert\cdot\rVert_*$) its norm,
$W^{k,p}(\Omega)$	the Sobolev space of functions whose distributional derivatives up to k^{th} order belongs to $L^p(\Omega)$, cf. (1.30) on p.15.
$W_0^{1,p}(\Omega)$	the Sobolev space of functions from $W^{1,p}(\Omega)$ whose traces on Γ vanish,
$W^{-1,p'}(\Omega)$	the dual space to $W_0^{1,p}(\Omega)$,
$W_{\mathrm{loc}}^{k,p}(\Omega)$	the set of functions v on Ω whose restrictions $v\lvert_O$, with any open O such that $\bar{O}\subset\Omega$, belong to $W^{k,p}(O)$,
$W^{1,p,q}$	the Sobolev space of abstract functions having the time-derivative, see (7.1) on p. 187,
$W^{1,p,\mathcal{M}}$	the Sobolev space of abstract functions whose derivatives are measures, see (7.40) on p. 196,
$W^{2,\infty,p,q}$	the Sobolev space of abstract functions having the second time-derivative, see (7.4) on p. 188,

$W^{1,p}_{0,\mathrm{div}}$	the set of divergence-free functions $v \in W^{1,p}_0(\Omega;\mathbb{R}^n)$, see (6.29) on p. 168,
Γ	the boundary of a domain Ω,
δ_K	the indicator function of a set K; i.e. $\delta_K(\cdot) = 0$ on K and $\delta_K(\cdot) = +\infty$ on the complement of K,
δ_x	the Dirac distribution (measure) supported at a point x,
Δ	the Laplace operator: $\Delta u = \mathrm{div}(\nabla u) = \frac{\partial^2}{\partial x_1^2}u + \cdots + \frac{\partial^2}{\partial x_n^2}u$,
Δ_p	the p-Laplace operator: $\Delta_p u = \mathrm{div}(\lvert\nabla u\rvert^{p-2}\nabla u)$ with $p > 1$,
ν	the unit outward normal to Γ at $x \in \Gamma$, $\nu = \nu(x)$,
Σ	the side surface of the cylinder Q, i.e. $I \times \Gamma$, or a σ-algebra of sets,
χ_S	the characteristic function of a set S; i.e. $\chi_S(\cdot) = 1$ on S and $\chi_S(\cdot) = 0$ on the complement of S,
Ω	a bounded, connected, Lipschitz domain, $\Omega \subset \mathbb{R}^n$,
$\bar{\Omega}$	the closure of Ω,
$\subset$	a subset, or a continuous embedding,
$\Subset$	a compact embedding,
$\int_\Omega \ldots \mathrm{d}x$	integration according to the n-dimensional Lebesgue measure,
$\int_\Gamma \ldots \mathrm{d}S$	integration according to the $(n-1)$-dimensional surface measure on Γ,
$\partial\Phi$	the subdifferential of the convex functional $\Phi : V \to \mathbb{R}$,
$(\cdot)^{-1}$	the inverse mapping,
$(\cdot)^*$	the dual space, see p.3, or the adjoint operator, see p.5, or the Legendre-Fenchel conjugate functional, see p.267,
$(\cdot)^+$, $(\cdot)^-$	the positive and the negative parts, respectively, i.e. $u^+ = \max(u, 0)$ and $u^- = \min(u, 0)$,
$(\cdot)'$	the Gâteaux derivative, cf. p.5, or a partial derivative, or the conjugate exponent, see p.12,
$(\cdot)\|_S$	the restriction of a mapping or a function on a set S,
$(\cdot)^\top$	the transposition of a matrix,
$\to$	a convergence (in a locally convex space) or a mapping between sets,
$\searrow$	convergence on $\mathbb{R}$ from the right; similarly $\nearrow$ means from the left,
$\rightrightarrows$	a set-valued mapping (e.g. $A : X \rightrightarrows Y$ abbreviates $A : X \to 2^Y$ = the set of all subsets of Y),
$\mapsto$	a mapping of elements into other ones, e.g. $A : u \mapsto f$ where $f = A(u)$,
∇	the spatial gradient: $\nabla u = (\frac{\partial}{\partial x_1}u, \ldots, \frac{\partial}{\partial x_n}u)$,
$\cdot$	a position of an unspecified variable, or the scalar product of vectors; i.e. $u \cdot v := \sum_{i=1}^m u_i v_i$ for $u, v \in \mathbb{R}^m$,
$:$	the scalar product of matrices; i.e. $A : B := \sum_{i=1}^n \sum_{j=1}^m A_{ij}B_{ij}$,
$:=$	the definition of a left-hand side by a right-hand-side expression,
$\otimes$	the tensorial product of vectors: $[u \otimes v]_{ij} = u_i v_j$,

$\langle\cdot,\cdot\rangle$	the bilinear pairing of spaces in duality, cf. p.3,
$\langle\cdot,\cdot\rangle_{\mathrm{s}}$	the semi-inner product in a Banach space, cf. (3.7) on p.91,
$(\cdot,\cdot)$	the inner (i.e. scalar) product in a Hilbert space, cf. (1.4) on p.2,
$\|\cdot\|$	a norm on a Banach space, see p.1,
$|\cdot|$	a seminorm on a Banach space, or an Euclidean norm in $\mathbb{R}^n$.