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Manfred Stern

Semimodular Lattices

Dedicated to Garrett Birkhoff

Springer Fachmedien Wiesbaden GmbH 1991

The book develops the theory of semimodular lattices with an emphasis on strong semimodular lattices of finite length and finite-modular AC-lattices of infinite length.

Das Buch entwickelt die Theorie der semimodularen Verbände unter besonderer Berücksichtigung der stark semimodularen Verbände endlicher Länge und der endlich-modularen AC-Verbände unendlicher Länge.

Le livre développe la théorie des treillis semimodulaires en mettant l'accent sur les treillis fortement semimodulaires de longueur finie et sur les AC-treillis finitement modulaires de longueur infinie.

В книге представлена теория полумодулярных структур с особым учетом сильно полумодулярных структур конечной длины и конечно-модулярных АС-структур бесконечной длины.

INTRODUCTION

The theory of lattices, initiated by Dedekind in the past century, and revived in the thirties by Garrett Birkhoff, F. Klein-Barmen, Ore, and von Neumann, is only in our time coming into its own.

The fledgling theory was handicapped by a contingent historical circumstance. The peculiarities of mathematical personality of the founders made lattice theory less welcome to the mathematical public of the time than it otherwise might have been. Thus Dedekind was widely thought in his time to be far too abstract for his own good, and some of his peers, notably Kronecker, did not hesitate to state their loud and clear disapproval. Later on, the tempers of Garrett Birkhoff and John von Neumann clashed with those of some of the "mainstream" mathematicians of their time.

Norman Levinson once related to me the following anecdote about von Neumann. Invited to deliver the weekly mathematics colloquium at Harvard sometime in the thirties, he chose the subject of his current interest, namely, continuous geometries. At the end of the lecture, as the public was streaming out, G. H. Hardy, who was at the time visiting Cambridge, was overheard whispering to G. D. Birkhoff (Garrett's father): "He is quite clearly a very brilliant man, but why does he waste his time on this stuff?"

I myself, when still an assistant professor, was once stopped in the hall of M. I. T. by a much senior colleague, who, staring at me in the eyes, demanded in no uncertain terms: "Admit! All lattice theory is trivial!"

The public's attitude towards lattice theory did not change until recently. It is interesting to speculate how much more readable Grothendieck's theory of schemes would be, if only their discoverer had been familiar with the rudiments of the theory of distributive lattices; similarly, expositions of ideal theory bearing on number theory would become far more transparent, if only their authors openly admitted that a great many arithmetic properties depend on whether the lattice of ideals of a number field is distributive or modular, and on purely latticial counts. But we shall have to wait until the next generation to see the clearer language of lattice theory adopted in algebra to the same extent as it has been adopted in combinatorics.

Contemporary lattice theory largely ranges between two equally attractive poles. At one end are ideas that originated with the distributive law, and which have filtered through logic, topos theory, and sundry other subjects. At the other end we find the heirs of the old "synthetic geometry", who have met an unexpected customer in combinatorics. The semimodular property, first observed in the theory of matroids, and more recently reappearing in a host of handsome theories now in full swing of development, has proved to be a central unifying concept, with no end of applications in sight. Theories that seemed to be the preserve of geometric lattices are gradually, and not without substantial effort, being carried over to the more difficult but far more inclusive class of semimodular lattices, to everybody's rejoycing.

Professor Stern's book is the first to include the latest developments in the theory of semimodular lattices. His clear presentation, and his readable mathematical style, will make his treatise an indispensable *vade mecum* for lattice theorists, and for all discrete mathematicians everywhere.

Gian-Carlo Rota

Cambridge, Massachusetts, November 12, 1990

PREFACE AND ACKNOWLEDGEMENTS

Lattice theory is a comparatively young branch of algebra. Although rooted in George Boole's attempt to formalize propositional logic, a systematic theory of lattices was developed much later, starting about 1930, notably through the efforts of Garrett Birkhoff, V. Glivenko, J. v. Neumann, O. Ore, S. MacLane and others. The concept of semimodularity was discovered by Garrett Birkhoff at that time; it emerged from the study of certain closure operators. Since then many results on semimodular lattices have appeared.

In the present text I have tried to incorporate some recent results into the well-known body of results.

The text consists of four chapters, the first three of which deal almost exclusively with lattices of finite length, whereas the fourth chapter deals mostly with one important class of semimodular lattices of infinite length. Throughout the text I have included sections (sometimes in a more or less informal way) to indicate connections with other topics and objects outside of lattice theory. Every chapter ends with a section "Further topics and remarks" briefly reviewing results not included in the main body of the text.

The aspects of semimodularity dealt with in more detail are largely motivated by Faigle's notion of a strong (semimodular) lattice (s. FAIGLE [1980 b]) and by the concept of a finite-modular AC-lattice as treated in the book "Theory of Symmetric Lattices" (cf. MAEDA-MAEDA [1970]). Accordingly, the aspects of semimodularity I deal with are of geometrical and combinatorial nature or have their roots in algebraic questions of functional analysis. In the following I give a short survey of the contents.

Chapter I reviews the place of semimodularity in lattice theory (Section 1), gives the most important examples (Section 2), and indicates the relationship between geometric lattices and matroids (Section 8), and also the relationship between semimodular lattices and greedoids (Section 9). A greedoid is a combinatorial object extending the notion of a matroid. Roughly speaking, (interval) greedoids are related to matroids in a similar way as finite semimodular lattices to geometric lattices. The most important characterizations of semimodularity are given in Section 4; some older ones without proof and some more recent ones with proofs. A "Butterfly Lemma" which reflects a property of semimodular lattices is proved in Section 10. Some standard arithmetical results are given in Section 5 to prepare for arithmetical questions dealt with later on, notably in Section 27 which considers the scope of the Kurosh-Ore replacement property in semimodular lattices of finite length.

Modular lattices of finite length and geometric lattices are the most important classes of semimodular lattices of finite length. I shall not deal with these special classes in detail. Occasionally, however, some pertinent results are proved or cited, mainly in the modular case. For instance, Section 6 gives a proof of the Kurosh-Ore theorem and the Schmidt-Ore theorem for modular lattices of finite length; a proof of Dilworth's covering theorem is presented in Section 7. In Sections 12 - 14 I show informally how to go from modular lattices via semimodular and supersolvable lattices to Cohen-Macaulay posets.

Chapter II considers the properties of strongness, consistence, and of being balanced, each of which may be viewed, like semimodularity as a generalization of modularity. Section 16 and 17 extend Faigle's concept of a strong join-irreducible element in two ways by introducing strong elements and strict elements. Section 18 relates strongness to a property which is called "lower balanced" (Section 18). Looking at meet-distributive lattices, we shall see that these are both strong and consistent, but not in general balanced. A further connection to objects outside of lattice theory is given by the so-called convex geometries; these are the combinatorial counterpart to meet-distributive lattices. On the other hand, convex geometries also give rise to special greedoids. This relationship is outlined in Section 21.

Chapter III combines semimodularity with the property of strongness. This leads to strong semimodular lattices which were first investigated by Faigle. Many properties of geometric lattices and of finite modular lattices can be proved in original or modified form for this broader class of lattices. I give some equivalent conditions for a semimodular lattice to be strong. These characterizations include a forbidden sublattice theorem, a Steinitz-MacLane exchange property, the Kurosh-Ore replacement property, join-symmetry, and basis exchange. It will also be shown that, in semimodular lattices of finite length, the concepts of strongness, consistence, and of being balanced all have the same meaning. After this I consider what I call generalized matroid lattices, a proper subclass of the strong semimodular lattices. Many important examples of strong semimodular lattices (notably geometric lattices and modular lattices of finite length) belong to this subclass. Something more can be shown if a strong semimodular lattice is cyclically generated, that is, if the principal ideal generated by each join-irreducible is a chain (Section 30). Section 31 gives a further instance of the greedy algorithm on a partially ordered set (namely on the poset of all join-irreducibles of a cyclically generated modular lattice of finite length).

Chapter IV first considers the interrelationship of the three most important concepts of semimodularity for lattices of infinite length. It then deals mainly with a particular class of semimodular lattices of infinite length; these are the atomistic lattices which satisfy the conditions of both upper and lower semimodularity as defined in Chapter I. In the terminology of MAEDA-MAEDA [1970], lattices of this type are called finite-modular AC-lattices. These lattices are, in general, neither finite nor modular. They occur, for instance, as lattices of closed subspaces of an infinite dimensional Hilbert space and are in this sense related to functional analysis. Section 36 presents results on finite-distributive AC-lattices, a subclass of the class of finite-modular AC-lattices. Section 37 next characterizes the property that each finite element (i. e. each join of finitely many atoms) of a finite-modular AC-lattice has a complement and Section 38 reviews the interrelationship of complementedness conditions. Section 39 extends results on complementedness which are known for modular lattices (of finite length), to broader classes of lattices. Section 40 considers static and biatomic lattices, mainly in the semimodular case. Finally Section 41 relates the ideal of the finite elements of an AC-lattice to the theory of standard ideals. This includes applications to atomistic Wilcox lattices, affine matroid lattices and to orthomodular AC-lattices.

Some important concepts used but not defined in the text are briefly explained in an Appendix. There are many figures (mainly Hasse-diagrams) throughout the text that illustrate definitions, examples, and proofs. We share the opinion of RIVAL-SANDS [1980] who wrote:

> *Pictorial aids to reasoning are generally ignored in scholarly mathematics ... After all, pictures may oversimplify and distort the real situation and mislead from our mission ... Admittedly, pictures in print are not feasible in many branches of mathematics. A happy exception is lattice theory.*

In every section the numbering of the formulae starts anew. References are given in the form "MAEDA [1977]" which refers to a paper (or preprint, or book) by Maeda published in 1977. Such references as "[1985 a]" or "[1985 b]" indicate that the Bibliography contains more than one item published or written by the author in that year. Aside from the sources referred to in the main body of the text I have tried to list in the Bibliography as many additional papers as possible on semimodularity "proper". In compiling the Bibliography' I turned to many people involved in some way or another in the subject matter and asked them, sometimes rather frequently, for information. I should like to thank all of them without addressing each of them individually.

An index and the table of notation indicate where a concept or symbol is first used or introduced.

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