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Cyclic Homology

With 24 Figures

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Une mathématique bleue
Dans une mer jamais étale
D'où nous remonte peu à peu
Cette mémoire des étoiles

. . .

Léo Ferré

À Éliane

Préface

Il y a maintenant 10 ans que l'homologie cyclique a pris son essor et le rythme de parution des publications à son sujet confirme son importance. Durant ce laps de temps l'effet de sédimentation a pu opérer et il devenait possible, sinon nécessaire, de disposer d'un ouvrage de référence sur le sujet.

Je n'ai pu écrire ce livre que grâce aux enseignements et à l'aide de nombreux collègues, que je voudrais remercier ici. Les cours de topologie algébrique d'Henri Cartan, qui resteront certainement dans la mémoire de ses auditeurs, ont constitué mon initiation et il est difficile d'en être digne. Max Karoubi m'a introduit à la K-théorie, topologique tout d'abord, puis algébrique ensuite, et son enseignement n'a pas peu contribué à ma formation. Dan Quillen a été constamment présent tout au long de ces années. Au début ce fut par ses écrits (cobordisme et groupes formels, homotopie rationnelle), puis par ses exposés (K-théorie algébrique) et, plus récemment, par une collaboration qui est à l'origine de ce livre. Les conversations et discussions avec Alain Connes furent toujours stimulantes et exaltantes. Ses encouragements et son aide furent pour moi un soutien constant. Je voudrais aussi remercier Zbignew Fiedorowicz, Claudio Procesi et Ronnie Brown pour leur collaboration efficace et amicale. Remerciements aussi à Keith Dennis pour m'avoir donné l'opportunité de faire un cours sur l'homologie cyclique à Cornell University au tout début de la rédaction et à Jean-Luc Brylinski pour un semestre fructueux passé à Penn State University. Ce livre doit aussi beaucoup à de nombreux autres collègues, soit pour des discussions, soit pour des commentaires pertinents, en particulier à L. Avramov, P. Blanc, J.-L. Cathelineau, C. Cuvier, S. Chase, P. Gaucher, F. Goichot, P. Julg, W. van der Kallen, C. Kassel, P. Ion, J. Lodder, R. McCarthy, A. Solotar, T. Pirashvili, C. Weibel et le rapporteur. Mamuka Jiblaze a relu entièrement le manuscrit durant la phase finale et je lui en sais gré.

Je voudrais aussi mentionner tout particulièrement Maria Ronco pour m'avoir toujours écouté avec attention, pour avoir lu plusieurs versions de ce livre et pour avoir corrigé de nombreuses imprécisions. Enfin et surtout je terminerai en remerciant chaleureusement Daniel Guin pour le nombre incalculable d'heures que nous avons passé ensemble devant un tableau noir et dont je garde le meilleur souvenir.

Par 48° 35′N et 7° 48′E, le 12 janvier 1992.

Contents

Introduction

Cyclic homology appeared almost simultaneously from several directions. In one, Alain Connes [C] developed cyclic homology as a non-commutative variant of the de Rham cohomology, in order to interpret index theorems for non-commutative Banach algebras, via a generalization of the Chern character. In another, cyclic homology was shown to be the primitive part of the Lie algebra homology of matrices by Boris Tsygan [1983], and by Dan Quillen and myself [1983, LQ]. This relationship shows that cyclic homology can be considered as a Lie analogue of algebraic K-theory, and, in fact, I met it for the first time through the cyclic property of some higher symbols in algebraic K-theory (cf. [Loday [1981]). There is still another framework where cyclic homology plays an important rôle: the homology of S^1-spaces, which provides the connection between index theorems and algebraic K-theory. We will see that cyclic homology theory illuminates a great many interactions between algebra, topology, geometry, and analysis.

The contents of the book can be divided into three main topics:

– cyclic homology of algebras (Chaps. 1–5), which essentially deals with homological algebra,

– cyclic sets and S^1-spaces (Chaps. 6–8), which uses the simplicial technique and some algebraic topology,

– Lie algebras and algebraic K-theory (Chaps. 9–11), which is about the relationship with the homology of matrices under different guises.

The last chapter (Chap. 12), which contains no proof, is essentially an opening towards Connes' work and recent results on the Novikov conjectures.

The cyclic homology of an algebra A consists of a family of abelian groups $HC_n(A)$, $n \geq 0$, which are, in characteristic zero, the homology groups of the quotient of the Hochschild complex by the action of the finite cyclic groups. This is the reason for the term "cyclic". The notation HC was for "Homologie de Connes", but soon became "Homologie Cyclique". This very first definition of Connes was slightly modified later on, so as to give a good theory in a charateristic-free context. In any case, the basic ingredient is the Hochschild complex, so the first chapter is about Hochschild homology, whose groups are denoted $HH_n(A)$, $n \geq 0$. Chapter 2 contains several definitions of cyclic homology, together with the basic properties of the functors HC_n. The most important one is Connes periodicity exact sequence,

$$\ldots \to HH_n(A) \to HC_n(A) \to HC_{n-2}(A) \to HH_{n-1}(A) \to \ldots \quad .$$

In Chap. 3 we perform some computation for tensor algebras, symmetric algebras, universal enveloping algebras and smooth algebras. We emphasize the relationship with the de Rham cohomology (in the commutative case). For smooth algebras, in characteristic zero, it takes the form of an isomorphism

$$HC_n(A) \cong \Omega^n_{A|k}/d\Omega^{n-1}_{A|k} \oplus H^{n-2}_{DR}(A) \oplus H^{n-4}_{DR}(A) \oplus \ldots \quad .$$

Chapter 4 is about the operations on cyclic homology: conjugation, derivation, product, coproduct, and λ-operations. These latter operations bring in some very interesting idempotents lying in the group algebra of the symmetric group, called the *Eulerian idempotents.* They are related to combinatorics (Eulerian numbers) and to the Campbell-Hausdorff formula. They permit us to show the existence of a λ-decomposition of the cyclic homology of a commutative algebra,

$$HC_n(A) = HC^{(1)}_n(A) \oplus \ldots \oplus HC^{(n)}_n(A) \; .$$

In both Chaps. 3 and 4 we give explicit isomorphisms and explicit homotopies (instead of using the acyclic model method) so as to give the possibility to extend these proofs to other settings (entire cyclic cohomology for instance).

In Chap. 5, important variations of cyclic homology are studied. The "negative cyclic homology", introduced by J.D.S. Jones and T. Goodwillie, is the right range for the Chern-Connes character. The "periodic cyclic theory" is close to the de Rham theory for commutative algebras. The "dihedral theory" comes in when dealing with skew-symmetric and symplectic matrices. We also study cyclic homology of differential graded algebras, since it is an efficient tool for computation.

The second part starts, in Chap. 6, with a detailed analysis of the relationship between the finite cyclic groups and the simplicial category Δ of non-decreasing maps on finite sets. It gives rise to the cyclic category ΔC of Connes. Other similar situations are studied for other families of groups: the dihedral groups, the symmetric groups, the hyperoctahedral groups, and the braid groups. The cyclic category permits us to interpret the cyclic groups as derived functors and to construct cyclic sets and cyclic spaces. The main point (Chap. 7) is that their geometric realizations are S^1-spaces and that, for any cyclic set X, there is an isomorphism

$$HC_*(k[X]) \cong H^{S^1}_*(|X|, k) \; .$$

An important example, which arises naturally by this procedure, is the free loop space of a topological space (equivalent to Witten's way of handling the free loop space of a manifold). We also include in this chapter the computation of the cyclic homology of a group algebra, which is going to play an important rôle in the construction of the Chern-Connes character. The study of this character is carried out in Chap. 8. The classical Chern character is a morphism from K-theory to de Rham cohomology. In the non-commutative framework the range space is cyclic homology; in fact negative cyclic homology is best. The construction of this Chern character

$$ch^- : K_n(A) \to HC_n^-(A) ,$$

was the main motivation of Connes in building the cyclic theory. This chapter ends up with an application to the idempotent conjecture.

The last part is essentially devoted to the relationship of the cyclic theory with homology of matrices, either (under their additive structure) Lie algebra homology, or (under their multiplicative structure) homology of the general linear group or more precisely algebraic K-theory. Chapter 9 is an account of the classical invariant theory used as a tool in Chap. 10. The main result of Chap. 10 claims that the homology of the Lie algebra of matrices is computable, in characteristic zero, in terms of cyclic homology (Loday-Quillen-Tsygan theorem),

$$H_*(gl(A)) \cong \Lambda(HC_{*-1}(A)) .$$

This result is supplemented with some partial results on the computation of $H_*(gl_r(A))$, r fixed. Conjectures (cf. 10.3.9) for the general case are proposed in terms of the λ-decomposition of $HC_*(A)$. Some variations are briefly treated: adjoint representation as coefficients, skew-symmetric and symplectic algebras. The last section introduces a completely new variant of Lie homology, called "non-commutative Lie algebra homology" and denoted $HL_n(\mathfrak{g})$, $n \geq 0$. It consists in replacing, in the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g}$ (used to define $H_n(\mathfrak{g})$, the exterior module $\Lambda\mathfrak{g}$ by the tensor module $T\mathfrak{g}$. The tricky point was to find the correct differential in this framework. Then, the analogue of the L-Q-T theorem mentioned above is

$$HL_*(gl(A)) \cong T(HH_{*-1}(A)) .$$

Important generalizations of this non-commutative theory, with Lie algebras replaced by groups or spaces, are to be expected.

Chapter 11 is devoted to algebraic K-theory and its relationship to cyclic homology. The first two sections form a short introduction to algebraic K-theory of rings. Then we study in detail the relationship between the K-theory of a nilpotent ideal I and the corresponding cyclic homology. The aim is to prove the following isomorphism, due to T. Goodwillie,

$$K_n(A, I) \otimes \mathbb{Q} \cong HC_{n-1}(A, I) \otimes \mathbb{Q} .$$

The rest of the chapter is a continuation of the chapter on the Chern character, with a succinct account of secondary characteristic classes as done by M. Karoubi.

We end this book with a chapter on "Non-Commutative Differential Geometry". The aim is to give an overview of some applications of the cyclic theory to the Godbillon-Vey invariant, to the index theorem for Fredholm modules, and to the Novikov conjecture on higher signatures and its K-theoretic analogue. This chapter is expository and without any proof. All these subjects are under active current research.

Among the five appendices the first four are recapitulations of notions, techniques and results used throughout the book. The last one, written by María Ofelia Ronco, is a survey, with proofs, on "smooth algebras".

Conceived as a comprehensive study of the cyclic homology theory, this book requires some acquaintance with homological algebra and for some chapters, some familiarity with the basic techniques of algebraic topology. However it is conceivable to give a graduate course in homological algebra from the first chapters or another one on the chapters on invariant theory and Lie algebras. We have tried to make the statements and the proofs as self-contained as possible, though at some particular points we refer to Cartan-Eilenberg [CE] or Mac Lane [ML] for details. Beginning with chapter one is not the only way to read this book. If one is only interested in the Lie algebra results, then one can go directly to Chap. 10 (or Chap. 9, if invariant theory is not at one's disposal). If one is interested in cyclic sets and S^1-spaces, then one can begin with Sects. 6.1 and 6.2, and then go directly to Chap. 7. For the construction of the Chern character, read Sects. 1.1, 2.1, 5.1, and then Chap. 8. More itineraries are possible, corresponding to other interests.

Most of the results are already in the literature, in research articles, though several proofs are original. The bibliographical comments at the end of each chapter try to give appropriate credit and information for further reading.

Notation and Terminology

The standard language and notation of set theory, homological algebra and algebraic topology is used throughout. For instance $\mathbb{Z}$ is the ring of integers, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, are the fields of rational, real and complex numbers respectively. The arrow $\hookrightarrow$ (resp. $\twoheadrightarrow$) stands for a monomorphism (resp. an epimorphism), that is an injective (resp. surjective) map if in the category of sets.

Categories are denoted by boldface characters : (**Sets**) for the category of sets, (**Spaces**) for the category of compactly generated spaces and continuous maps, (k-**Mod**) for the category of k-modules and k-linear maps, etc.

A notation like $\pi_n(X) := [S^n, X]$ indicates a definition of the left-hand term.

Throughout the book k denotes a commutative ring, which sometimes satisfies some conditions like k contains $\mathbb{Q}$ or k is a field. Every module M over k is supposed to be symmetric and unital: $\lambda m = m\lambda$, $1m = m$. An algebra A over k need not have a unit. If it has a unit, then it is called *unital.* The term "k-linear map" is often abbreviated into "map". Tensor products are taken over k unless otherwise stated, and so $\otimes = \otimes_k$.

The automorphism group of the set $\{1, 2, \ldots, n\}$ is called a *permutation group* and denoted by S_n. It is sometimes helpful to make it act on the set $\{0, 1, \ldots, n-1\}$ instead. The sign of a permutation $\sigma \in S_n$ is denoted by $\mathrm{sgn}(\sigma) \in \{\pm 1\}$.

For any discrete group G the group algebra $k[G]$ is the free module over k with basis G. On elements of G the product is given by the grouplaw. For other elements it is extended by linearity.

More notation is introduced in Sect. 1.0 and in the appendices A, B and C.

The symbol $\square$ indicates the end or the absence of a proof.

Standing assumption valid for the whole chapter or section are indicated in the introduction of the relevant chapter or section.

The exercises are, most of the time, interesting results that we want to mention, but do not prove. Hints or, more often, bibliographical references are given in brackets.