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# Nevanlinna's Theory of Value Distribution

## The Second Main Theorem and its Error Terms

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# Preface

The Fundamental Theorem of Algebra says that a polynomial of degree $d$ in one complex variable will take on every complex value precisely $d$ times, provided the values are counted with their proper multiplicities. Toward the end of the nineteenth century, Picard [Pic 1879] generalized the Fundamental Theorem of Algebra by proving that a transcendental entire function – a sort of polynomial of infinite degree – must take on all but at most one complex value infinitely many times. However, there are a great many infinities, and after Picard's work, mathematicians tried to distinguish between "different" infinite degrees.

In hindsight, one recognizes that the key to progress was viewing the degree of a complex polynomial $P(z)$ as the rate at which the maximum modulus of $P(z)$ approaches infinity as $|z| \to \infty$. A decade after Picard's theorem, work of J. Hadamard [Had 1892a], [Had 1892b], [Had 1897] proved there was a strong connection between the growth order of an entire function and the distribution of the function's zeros. E. Borel [Bor 1897] then proved a connection between the growth rate of the maximum modulus of an entire function and the asymptotic frequency with which it must attain all but at most one complex value. Finally, R. Nevanlinna [Nev(R) 1925] found the right way to measure the "growth" of a meromorphic function and developed the theory of value distribution which now bears his name, and which is the subject of this book.

In the late 1920's, Nevanlinna organized his theory into the monograph [Nev(R) 1929]. That theory, including more recent developments, also makes up a substantial part of his his later monograph [Nev(R) 1970]. Another classic monograph, unfortunately now out of print, that discusses the theory in detail is that of W. Hayman [Hay 1964]. Many of those who grew up in the Russian speaking part of the world learned the theory from the excellent book by A. A. Gol'dberg and I. V. Ostrovskiĭ [GoOs 1970], a book which for some reason has yet to be translated into English. One of the reasons we have decided to write the present book is that the above mentioned books, though far from obsolete, are now somewhat out of date, and with the exception of [Nev(R) 1970], are becoming harder to find. We hasten to point out though that each of the above books contains many interesting topics and ideas we do not touch on here, and that we definitely do not see our present work as a replacement for any of these "classics."

R. Nevanlinna's original proof [Nev(R) 1925] of his "Second Main Theorem," the focus of our book, makes heavy use of special properties of logarithmic derivatives. Almost immediately after R. Nevanlinna proved the Second Main Theorem, F. Nevanlinna, R. Nevanlinna's brother, gave a "geometric" proof of the Second Main Theorem – see [Nev(F) 1925], [Nev(F) 1927], and the "note" at the end of [Nev(R) 1929]. This geometric viewpoint was later expanded upon by T. Shimizu [Shim 1929] and L. Ahlfors [Ahlf 1929], [Ahlf 1935a], and still later by S.-S. Chern [Chern 1960]. Both the "geometric" and "logarithmic derivative" approaches to the Second Main Theorem has certain advantages and disadvantages, so we will treat both approaches in detail.

Despite the geometric investigation and interpretation of the main terms in Nevanlinna's theorems, the so-called "error term" in Nevanlinna's Second Main Theorem was, until recently, largely ignored. Motivated by an analogy between Nevanlinna theory and Diophantine approximation theory, discovered independently by C. F. Osgood [Osg 1985]and P. Vojta [Vojt 1987], S. Lang recognized that the careful study of the error term in Nevanlinna's Second Main Theorem would be of interest in itself. To promote its study, Lang wrote the lecture notes volume [Lang 1990], where in the introduction he wrote:

> "... it seemed to me useful to give a leisurely exposition which might lead people with no background in Nevanlinna theory to some of the basic problems which now remain about the error term. The existence of these problems and the possibly rapid evolution of the subject ... made me wary of writing a book, ... "

As Lang predicted, a flurry of activity in the investigation of the error term, much of it involving the second author of this book, developed after [Lang 1990] appeared. Now things have calmed down on this front, and the error term in Nevanlinna's Second Main Theorem is very well understood and full of interesting geometric meaning. What remains to be done probably requires fundamentally new ideas, and so we felt this was a good time to write a book collecting together the existing work on error terms.

We have included a small sampling of applications for Nevanlinna's theory because we feel some exposure for the reader to the myriad of possible applications is essential to the reader's aesthetic appreciation of the field. However, applications are not the emphasis of this work, and indeed, many of the applications we discuss are due to Nevanlinna himself and already present in his 1929 monograph [Nev(R) 1929]. We do describe in some detail how knowledge of explicit error terms in Nevanlinna's theory provides a new look at some old theorems, but we have left it to others to write up to date accounts of applications to such subjects as differential equations, complex dynamics, and unicity theorems. For a more in depth overview of applications, the reader may want to read the book of Jank and Volkmann [JaVo 1985], and those particularly interested in complex differential equations may want to look at I. Laine's book [Laine 1993].

Because the connection between Nevanlinna theory and Diophantine approximation has been the motivation for so much of the current work in Nevanlinna theory and because of the considerable cross-fertilization between the two fields, we have included several sections discussing this connection in some detail, although lacking complete proofs. These sections assume no background at all in number theory, and we hope these sections will whet the appetites of a few die hard analysts enough that they will seek out more advanced treatments such as [Vojt 1987].

While including the state of the art in error terms, we have retained Lang's philosophy of providing a leisurely introduction to the field to those with no background in Nevanlinna theory, and this book will be easily accessible to those with only a basic course in one complex variable. Some readers may feel a bit uncomfortable at times because we did not hesitate in our use of the language of differential forms. Readers having difficulty with this language may want to consult a book such as [Spiv 1965]. We have two reasons for using this language of differential forms. First, we believe it is the language that most clearly and efficiently conveys the ideas behind some of the geometric arguments, and second, learning this language is absolutely essential to learning the several variable theory, a topic we plan to address in a future volume.

Finally, the reader should be aware that our bibliography is by no means a complete guide to the literature in the vast field of value distribution theory. Rather, we have tried to cite references that give the reader a good sense of the historical origins of the main ideas around the Second Main Theorem, and we have tried to provide a guide to the most recent work on the Second Main Theorem's error terms. We made no attempt to provide a complete history of each idea from its birth to today's state of the art error terms. Thus, our bibliography omits many important works that have advanced the field over the years.

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# A Word on Notation

We use the standard notation of $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$ to denote the integers, real numbers, and complex numbers, respectively. We use the notation $\mathbf{Z}_{>0}$ to denote those integers that are $> 0$. We use $\mathbf{C}^n$, $\mathbf{R}^n$, *etc.* to denote the $n$-th Cartesian products of these spaces.

The closed interval $\{x \in \mathbf{R} : a \leq x \leq b\}$ is denoted $[a,b]$, and the open interval is denoted $(a,b)$. Half-open intervals are denoted $(a,b]$ or $[a,b)$.

A function is called $C^\infty$ if derivatives of all orders exist. If $f$ is a function of several variables, then $C^\infty$ means all partial derivatives exist. A function is called $C^k$ if all (partial) derivatives of order $\leq k$ exist and are continuous.

If $X$ is a set in $\mathbf{R}^n$, in $\mathbf{C}^n$, or in some other space, we denote the closure of $X$ by $\overline{X}$.

We write complex numbers either as $z = x + yi$ or $z = x + y\sqrt{-1}$. As we often want to keep the letter $i$ free for summation indices, we tend use the $\sqrt{-1}$ notation, though the $i$ notation looks better in exponents. If $z = x + y\sqrt{-1}$ is a complex number, then we use $\bar{z} = x - y\sqrt{-1}$ to denote the complex conjugate. The real part $x$ is denoted $\operatorname{Re} z$, and the imaginary part $y$ is denoted $\operatorname{Im} z$. Note that if $X$ is a multi-point subset of $\mathbf{C}$, then $\overline{X}$ will be used to denote the closure of $X$ in $\mathbf{C}$ as defined in the last paragraph, and it does *not* denote the image of $X$ under complex conjugation.

We often write $f \equiv 0$ when we want to say that a function $f$ is the zero function, and similarly the notation $f \not\equiv 0$ means that $f$ is not the constant function $0$. Purists will argue that the proper notation for this is $f = 0$ and $f \neq 0$ respectively, but it sometimes happens that some authors write $f(z) \neq 0$ to mean $f$ never takes on the value $0$, whereas others use this to mean $f$ does not vanish at the point $z$. Thus it is convenient to have the notation $f \not\equiv 0$ to avoid this confusion.

As do the physicists, we use the symbols $\ll$ and $\gg$ to mean "much less than" and "much greater than." Thus, $r \gg 0$, means for all $r$ sufficiently large.

We use big and little "*oh*" notation throughout for asymptotic statements, though in a slightly non-standard way. For example, if $f(t)$ and $g(t)$ are real-valued functions of a real variable $t$ and if $h(t)$ is a real-valued function which is positive as $t \to t_0$, meaning that $h(t)$ is positive for $t$ sufficiently near $t_0$ (or sufficiently large if $t_0 = \infty$), then if we write $f(t) \leq g(t) + O(h(t))$, and we are interested in what happens asymptotically as $t \to t_0$, then we mean that

$$\limsup_{t \to t_0} \frac{f(t) - g(t)}{h(t)} < \infty.$$

When we write $f(t) \leq g(t) + o(h(t))$, then we mean

$$\limsup_{t \to t_0} \frac{f(t) - g(t)}{h(t)} \leq 0.$$

Usually when we write such asymptotics, it will be clear from the context what asymptotic value $t_0$ we are interested in, and we will not state this explicitly. Most often, we will mean as $t \to \infty$. If we write $f(t) = g(t) + O(h(t))$, then we mean that both $f(t) \leq g(t) + O(h(t))$ and $g(t) \leq f(t) + O(h(t))$, and similarly for the little "*oh*" notation. Often $h(t)$ will be taken to be the constant function 1. So for example, the statement $f(t) \leq g(t) + O(1)$, means that $f(t) - g(t)$ is bounded above for $t$ sufficiently near $t_0$ (sufficiently large in the typical case when $t_0 = \infty$). Occasionally, we may write something like $f(t) \leq g(t) + O_\varepsilon(h(t))$ to emphasize that

$$\limsup_{t \to t_0} \frac{f(t) - g(t)}{h(t)}$$

is bounded by a constant that may depend on some external parameter $\varepsilon$.

Constants which do not especially interest us will often be called $C$ or some other unfancy name. Thus, the symbol $C$ stands for many different constants throughout the book, and may even change its meaning within a single proof. Constants that we find interesting or want to refer back to at a later point are given names with subscripts, such as $c_{\mathrm{fmt}}$ or $\beta_1$. All such "long-term" constants appear in the Glossary of Notation so that their definitions can be easily located.

Theorems, lemmas, propositions, and so forth are numbered by chapter and section. Thus, Theorem 2.3.1 would refer to the first theorem in the third section of Chapter 2.

Especially important equations are also numbered by chapter and section. Equations that we want to refer back to several times within the context of a single proof, but never again later, are labeled by $(*)$, $(**)$, *etc.* Thus, there are many equation $(*)$'s, and a reference to equation $(*)$ always refers to the most recently occurring equation $(*)$.

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