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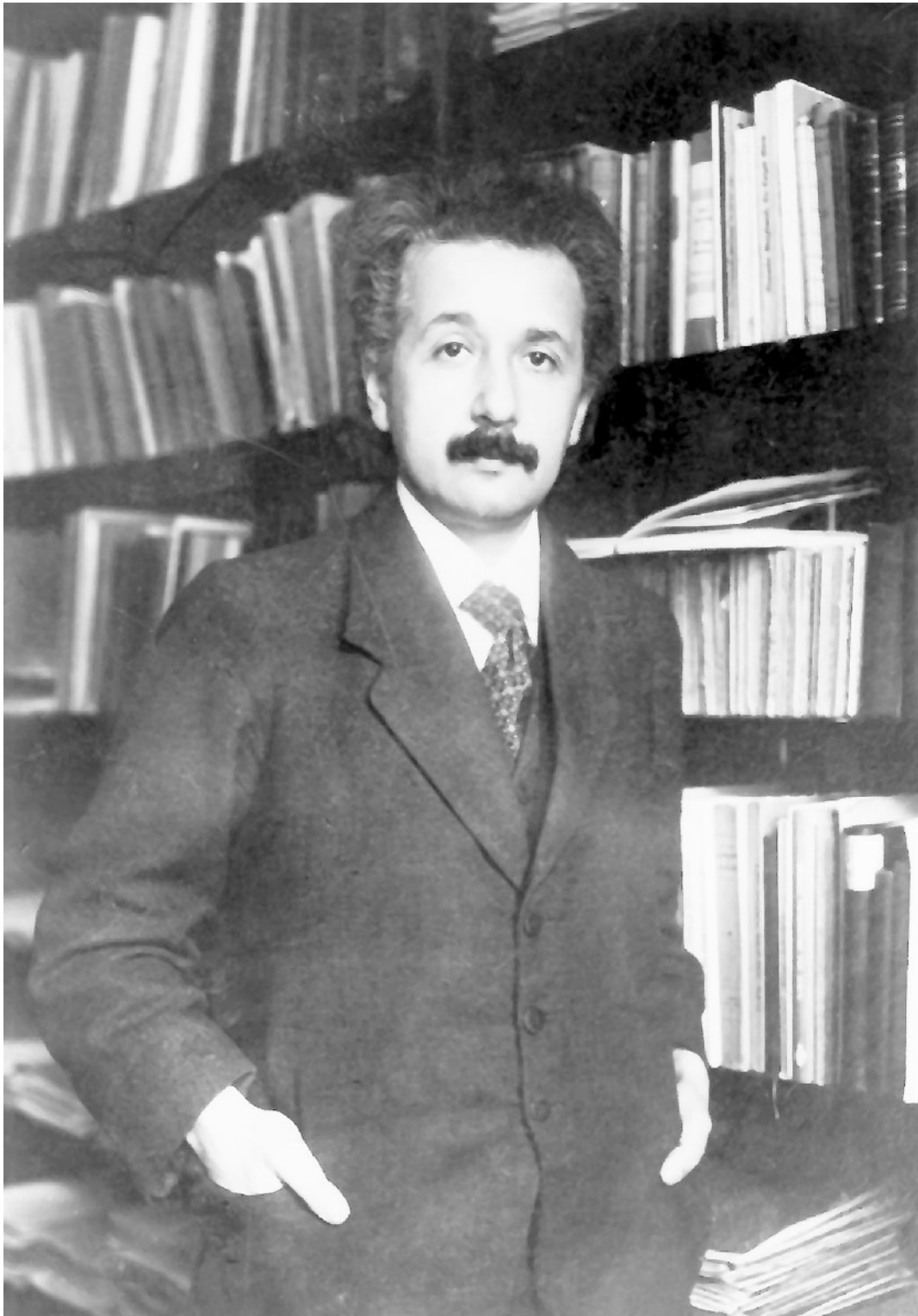
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Albert Einstein (1879–1955), in his study, Berlin 1916.
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Norbert Straumann

General Relativity

With Applications to Astrophysics

With 111 Figures



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Preface

Physics and mathematics students are as eager as ever to become acquainted with the foundations of general relativity and some of its major applications in astrophysics and cosmology. I hope that this textbook gives a comprehensive and timely introduction to both aspects of this fascinating field, and will turn out to be useful for undergraduate and graduate students.

This book is a complete revision and extension of my previous volume *‘General Relativity and Relativistic Astrophysics’* that appeared about twenty years ago in the Springer Series *‘Texts and Monographs in Physics’*; however, it cannot be regarded just as a new edition.

In Part I the foundations of general relativity are thoroughly developed. Some of the more advanced topics, such as the section on the initial value problem, can be skipped in a first reading.

Part II is devoted to tests of general relativity and many of its applications. Binary pulsars – our best laboratories for general relativity – are studied in considerable detail. I have included an introduction to gravitational lensing theory, to the extent that the current literature on the subject should become accessible. Much space is devoted to the study of compact objects, especially to black holes. This includes a detailed derivation of the Kerr solution, Israel’s proof of his uniqueness theorem, and a derivation of the basic laws of black hole physics. Part II ends with Witten’s proof of the positive energy theorem.

All the required differential geometric tools are developed in Part III. Readers who have not yet studied a modern mathematical text on differential geometry should not read this part in linear order. I always indicate in the physics parts which mathematical sections are going to be used at a given point of the discussion. For example, Cartan’s powerful calculus of differential forms is not heavily used in the foundational chapters. The mathematical part should also be useful for other fields of physics. Differential geometric and topological tools play an increasingly important role, not only in quantum field theory and string theory, but also in classical disciplines (mechanics, field theory).

A textbook on a field as developed and extensive as general relativity must make painful omissions. When the book was approaching seven hundred pages, I had to give up the original plan to also include cosmology. This is

really deplorable, since cosmology is going through a fruitful and exciting period. On the other hand, this omission may not be too bad, since several really good texts on various aspects of cosmology have recently appeared. Some of them are listed in the references. Quantum field theory on curved spacetime backgrounds is not treated at all. Other topics have often been left out because my emphasis is on direct physical applications of the theory. For this reason nothing is said, for instance, about black holes with hair (although I worked on this for several years).

The list of references is not very extensive. Beside some useful general sources on mathematical, physical, and historical subjects, I often quoted relatively recent reviews and articles that may be most convenient for the reader to penetrate deeper into various topics. Pedagogical reasons often have priority in my restrictive selection of references.

The present textbook would presumably not have been written without the enduring help of Thiemo Kessel. He not only typed very carefully the entire manuscript, but also urged me at many places to give further explanations or provide additional information. Since – as an undergraduate student – he was able to understand all the gory details, I am now confident that the book is readable. I thank him for all his help.

I also thank the many students I had over the years from the University of Zurich and the ETH for their interest and questions. This feedback was essential in writing the present book. Many of the exercises posed in the text have been solved in my classes.

Finally, I would like to thank the Tomalla Foundation and the Institute for Theoretical Physics of the University of Zurich for financial support.

Post Script I would be grateful for suggestions of all kind and lists of mistakes. Since I adopted – contrary to my habits – the majority convention for the signature of the metric, I had to change thousands of signs. It is unlikely that no sign errors remained, especially in spinorial equations.

Comments can be sent to me at [<norbert@physik.unizh.ch>](mailto:norbert@physik.unizh.ch).

Zurich, January 2004

Norbert Straumann

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