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Hardy Operators, Function Spaces and Embeddings

With 6 Figures

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Preface

Classical Sobolev spaces, based on Lebesgue spaces on an underlying domain with smooth boundary, are not only of considerable intrinsic interest but have for many years proved to be absolutely indispensable in the study of partial differential equations and variational problems. The embedding theorems and inequalities which feature so large in first courses on function spaces are key ingredients of the proofs of existence and regularity for elliptic boundary-value problems. There have been many developments of the basic theory since its inception, and of these we distinguish two which seem to us to be of particular interest:
(i) the consequences of working on space domains with irregular boundaries;
(ii) the replacement of Lebesgue spaces by more general Banach function spaces.

Both of these arise in response to demands imposed by concrete problems. For example, the ubiquitous nature of sets with fractal boundaries makes it unnecessary to give an extended justification of (i), while (ii) is very natural when faced with (degenerate) elliptic problems in which the coefficients of the differential operator satisfy more refined conditions than in classical situations. It is to be expected that these aspects of the theory will enjoy substantial further growth, but nevertheless we believe that the present state of affairs makes it desirable to have a connected account of those parts which seem to us to have reached a degree of maturity. This book is intended to do just that. Its main themes are Banach function spaces and spaces of Sobolev type based on them, especially when the space domain involved is a so-called generalised ridged domain; integral operators of Hardy type on intervals and on trees; and the distribution of the approximation numbers (singular numbers in the Hilbert space case) of embeddings of Sobolev spaces based on generalised ridged domains. A prerequisite for reading it is a good graduate course in real analysis.

Chapter 1 contains a variety of results and concepts which will be useful throughout the remainder of the book. It is for consultation, to be looked at when necessary. The next chapter is largely concerned with mappings T :

$L_p(a,b) \to L_q(a,b)$, where $1 \le p, q \le \infty$ and $-\infty < a < b \le \infty$, given by

$$(Tf)(x) = v(x) \int_a^x u(t)f(t)dt, \; x \in (a,b), \; f \in L_p(a,b).$$

Here u and v are given functions which satisfy certain integrability conditions. These operators are said to be of Hardy type, the original Hardy operator being given by taking $u = v = 1$ and $p = q$. They appear in a very natural way in connection with embeddings of Sobolev spaces based on the particular class of generalised ridged domains, as we shall see in Chapters 5 and 6; they also are of importance in some 'small ball' problems in probability theory, for which we refer to [157]. We give criteria for T to be bounded and determine its measure of non-compactness. This enables us to provide necessary and sufficient conditions for T to be compact. In the compact case, we furnish upper and lower bounds for the approximation numbers $a_n(T)$ of T which, when $p = q \in (1, \infty)$, lead to the interesting asymptotic result that

$$\lim_{n \to \infty} n a_n(T) = \frac{\gamma_p}{2} \int_a^b |u(t)v(t)| \, dt,$$

where $\gamma_p = \frac{1}{\pi} p^{1/p'} (p')^{1/p} \sin(\pi/p)$, $p' = p/(p-1)$, under appropriate conditions on u and v Further refinement of this is possible, giving remainder estimates for the approximation numbers. Finally, we show that many of these results can be taken over to the situation in which the interval (a,b) is replaced by a tree: this will be of crucial importance when we come in Chapters 5 and 6 to deal with embeddings of Sobolev spaces when the underlying subset of $\mathbb{R}^n$ is a generalised ridged domain. Amongst the byproducts of our analysis is the result (originally proved in [68] and [87]) that if (a,b) is a bounded interval in $\mathbb{R}$ and $1 < p < \infty$, then the approximation numbers $a_m(E_0)$ of the embedding E_0 of the Sobolev space $\overset{\circ}{W}{}_p^1(a,b)$ in $L_p(a,b)$ are given by

$$a_m(E_0) = \frac{\gamma_p}{2m}(b-a) \quad (m \in \mathbb{N}).$$

The precision of this owes much to the one-dimensional nature of the underlying domain (a,b), and is in marked contrast to known results (see, for example, [74]) concerning the approximation numbers of Sobolev embeddings of spaces based on open subsets of $\mathbb{R}^n$, which typically give sharp upper and lower bounds for these numbers but do not establish a genuine asymptotic behaviour.

In Chapter 3 we give an account of Banach function spaces and spaces of Sobolev type based on them. Our object here is to present, in a systematic way, some of the refinements of the classical Sobolev embedding theorems which have become known in the last ten years or so and which result from the use of scales of spaces which can be more finely tuned than the Lebesgue family. Originally, many of these results were proved for Sobolev spaces based on Lorentz-Zygmund or generalised Lorentz-Zygmund spaces rather than on

L_p, as in the classical situation. The strategy which we adopt, however, is to use Lorentz-Karamata spaces, which depend on the notion of slowly varying functions. By this means we are able to give a unified approach to the subject which is both more economical than a succession of ad hoc arguments used for particular circumstances and also helps to clarify the nature of the arguments which are deployed. As an illustration of the kind of result which we establish, consider the famous example involving the Sobolev space $W_n^1(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ $(n > 1)$ with smooth boundary. It is very well known that this space can be embedded in every $L_p(\Omega)$ space with $p \in (1, \infty)$, that it cannot be embedded in $L_\infty(\Omega)$, but that it can be embedded in an Orlicz space of exponential type, with Young function given by $\exp(t^{n/(n-1)}) - 1$. It turns out that if instead of using $L_n(\Omega)$ as the base for this Sobolev space we use an appropriate nearby Zygmund space, then the target space can be an Orlicz space of multiple exponential type.

Chapter 4 provides a discussion of Poincaré inequalities in the general setting of spaces $W(X, Y)$ of Sobolev type: here X and Y are Banach function spaces on a bounded domain Ω in $\mathbb{R}^n$, while $W(X, Y)$ is the set of all $f \in X$ with distributional gradient in Y. Connections are made between the Poincaré inequality and the measure of non-compactness of the embedding of $W(X, Y)$ in X, and numerous illustrative examples are given, including the cases in which X and Y are of Lebesgue or Orlicz type. The chapter contains a treatment of classical Sobolev and Poincaré inequalities under very weak conditions on Ω, such as that it should be a John domain. There is some discussion of the bewildering array of weak conditions on domains which can be found in the literature. To conclude we deal with the higher-dimensional analogue of the Hardy inequality handled in Chapter 2, again under quite weak conditions on the underlying space domain.

The generalised ridged domains which form the subject of Chapter 5 are certain domains in $\mathbb{R}^n$ which, roughly speaking, have a central axis, the so-called generalised ridge, which is the image in the domain of a tree under a Lipschitz map. This is a wide class of domains, and includes such diverse sets as horns, spirals, 'rooms and passages' and even snowflakes. There are two problems, in particular, addressed in this chapter: for Banach function spaces X, Y, Z defined on a generalised ridged domain Ω, what target space Z is permissible in an embedding $W(X, Y) \hookrightarrow Z$, and when is there a valid Poincaré inequality associated with the embedding $W(X, Y) \hookrightarrow X$? The second problem is shown to be related to the range of values of the measure of non-compactness of the embedding, a result originally due to Amick [6] when $X = Y = L_2$. Furthermore, it is proved that Poincaré-type inequalities yield the measure of non-compactness as a limit along a filter base of subsets of Ω, which makes precise the fact that any lack of compactness of the embedding is due to the singular nature of the set at the intersection of the boundary of Ω and the generalised ridge. The significance of generalised ridged domains stems from the fact that these problems can be reduced to corresponding ones on the associated trees, which, in certain cases, are simply intervals. This 'one-dimensionalisation' of the problems is a considerable advantage, and it is precisely this that makes tractable the problems which we study. Chapter 6 deals with the approximation numbers of Sobolev embeddings when the underlying space domain is a generalised ridged domain, and gives upper and lower estimates for these numbers. A point of interest is that we provide an L_p version of the Dirichlet-Neumann bracketing technique which is so familiar and effective in the L_2 theory of eigenvalues of elliptic operators.

Chapters are divided into sections, and some sections into subsections; the standard decimal classification is used. All chapters except the first contain a 'Notes and Remarks' section at their end, in which we provide supplementary information. Although we have not made a serious attempt to go into the history of the results given, we hope that the list of references will not be regarded as too exiguous and believe that the reader interested in historical matters will find it to be of some help. In addition to the bibliography, there is a glossary of terms and notation, together with author and subject indexes.

Brighton, Cardiff, *David E. Edmunds,*
June 2004 *W. Desmond Evans*

Contents

Basic symbols

$B(x,r)$: open ball in $\mathbb{R}^n$, centre x and radius r.
$\mathbb{N}$: natural numbers.
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
$\mathbb{Z}$: integers.
$\mathbb{R}$: real numbers.
$\mathbb{C}$: complex numbers.
$A \subset B$: A contained in B, or possibly equal to B.
$A \subset\subset B$: the closure of A is compact and is contained in B.
Ω: open subset of $\mathbb{R}^n$; Ω is a domain if it is also connected.
$\partial\Omega$: boundary of Ω.
$\overline{\Omega}$: closure of Ω.
$\mu_n = |\cdot|_n$: n-dimensional Lebesgue measure.
$|\Omega|$: Lebesgue n-measure of $\Omega \subset \mathbb{R}^n$.
$d(x,F)$: distance of x from set F; also written as $d_F(x)$ and $d(x)$.
χ_E: characteristic function of E.
$D_i u = \partial u / \partial x_i$.
$D^\alpha u = \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ if $\alpha = (\alpha_1, \cdots, \alpha_n)$, each $\alpha_j \in \mathbb{N}_0, |\alpha| = \alpha_1 + \cdots + \alpha_n$.
$\delta_+ = \max(\delta, 0)$.
$\|\cdot|X\|$: norm or quasi-norm on X.
$\ker(f)$: kernel of f.
$C^k(\Omega), k \in \mathbb{N}_0$: complex-valued functions f such that $D^\alpha f$ is continuous on Ω for $0 \le |\alpha| \le k$; denoted by $C(\Omega)$ for $k=0$.
$C^\infty(\Omega) = \bigcap_{k=0}^\infty C^k(\Omega)$.
$C^\lambda(\overline{\Omega}), \lambda \in (0,1]$: Hölder-continuous functions of exponent λ on $\overline{\Omega}$.
$C_0^k(\Omega)$: functions in $C^k(\Omega)$ with compact supports in Ω.
$C_0^\infty(\Omega) = \bigcap_{k=0}^\infty C_0^k(\Omega)$.
$L_p(\Omega)$: Lebesgue space of functions f with $|f|^p$ integrable on Ω if $0 < p < \infty$, ess sup u $< \infty$ if $p = \infty$.
$L_{p,loc}(\Omega)$: functions in $L_p(K)$ for every compact subset K of Ω.
$l_p(\mathcal{I})$: space of sequences $\{x_i\}_{i\in\mathcal{I}}$, $x_i \in \mathbb{C}$, such that $\|\{x_i\}|l_p(\mathcal{I})\| < \infty$, where

$$\|\{x_i\}|l_p(\mathcal{I})\| = \begin{cases} \left(\sum_{i\in\mathcal{I}} |x_i|^p\right)^{1/p}, & \text{for } 1 \le p < \infty, \\ \sup_{i\in\mathcal{I}} |x_i|, & \text{for } p = \infty. \end{cases}$$

$\omega_n = \pi^{n/2} / \Gamma(1 + \frac{n}{2})$: volume of unit ball in $\mathbb{R}^n$.
$c \lesssim d, d \gtrsim c$: c is bounded above by a multiple of d, the multiple being independent of any variables in c and d.
$c \approx d : c \lesssim d$ and $d \lesssim c$.
$X \hookrightarrow Y$: X is continuously embedded in Y.
$X \hookrightarrow\hookrightarrow Y$: X is compactly embedded in Y.
$f * g$: convolution of f and g.