

Martin Aigner
Günter M. Ziegler

Proofs from THE BOOK

Second Edition

Springer-Verlag Berlin Heidelberg GmbH

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Günter M. Ziegler

Proofs from **THE BOOK**

Second Edition

With 240 Figures
Including Illustrations
by Karl H. Hofmann



Springer

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Library of Congress Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek – CIP-Einheitsaufnahme

Aigner, Martin:

Proofs from the book / Martin Aigner; Günter M. Ziegler. Incl. ill. by Karl H. Hofmann. – 2. ed. –
Berlin; Heidelberg; New York; Barcelona; Hong Kong; London; Milan; Paris; Singapore;
Tokyo: Springer, 2001

Mathematics Subject Classification (2000): 00-01 (General)

ISBN 978-3-662-04317-2

ISBN 978-3-662-04315-8 (eBook)

DOI 10.1007/978-3-662-04315-8

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© Springer-Verlag Berlin Heidelberg 2001

Originally published by Springer-Verlag Berlin Heidelberg New York in 2001

Softcover reprint of the hardcover 2nd edition 2001

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Typeset in L^AT_EX by the authors

Cover design: de'blik, Berlin

Printed on acid-free paper SPIN 10875342 46/3142db – 5 4 3 2 1 0

Preface

Paul Erdős liked to talk about The Book, in which God maintains the perfect proofs for mathematical theorems, following the dictum of G. H. Hardy that there is no permanent place for ugly mathematics. Erdős also said that you need not believe in God but, as a mathematician, you should believe in The Book. A few years ago, we suggested to him to write up a first (and very modest) approximation to The Book. He was enthusiastic about the idea and, characteristically, went to work immediately, filling page after page with his suggestions. Our book was supposed to appear in March 1998 as a present to Erdős' 85th birthday. With Paul's unfortunate death in the summer of 1996, he is not listed as a co-author. Instead this book is dedicated to his memory.

We have no definition or characterization of what constitutes a proof from The Book: all we offer here is the examples that we have selected, hoping that our readers will share our enthusiasm about brilliant ideas, clever insights and wonderful observations. We also hope that our readers will enjoy this despite the imperfections of our exposition. The selection is to a great extent influenced by Paul Erdős himself. A large number of the topics were suggested by him, and many of the proofs trace directly back to him, or were initiated by his supreme insight in asking the right question or in making the right conjecture. So to a large extent this book reflects the views of Paul Erdős as to what should be considered a proof from The Book.

A limiting factor for our selection of topics was that everything in this book is supposed to be accessible to readers whose backgrounds include only a modest amount of technique from undergraduate mathematics. A little linear algebra, some basic analysis and number theory, and a healthy dollop of elementary concepts and reasonings from discrete mathematics should be sufficient to understand and enjoy everything in this book.

We are extremely grateful to the many people who helped and supported us with this project — among them the students of a seminar where we discussed a preliminary version, to Benno Artmann, Stephan Brandt, Stefan Felsner, Eli Goodman, Torsten Heldmann, and Hans Mielke. We thank Margrit Barrett, Christian Bressler, Ewgenij Gawrilow, Michael Joswig, Elke Pose, and Jörg Rambau for their technical help in composing this book. We are in great debt to Tom Trotter who read the manuscript from first to last page, to Karl H. Hofmann for his wonderful drawings, and most of all to the late great Paul Erdős himself.



Paul Erdős



"The Book"

Preface to the Second Edition

The first edition of this book got a wonderful reception. Moreover, we received an unusual number of letters containing comments and corrections, some shortcuts, as well as interesting suggestions for alternative proofs and new topics to treat. (While we are trying to record *perfect* proofs, our exposition isn't.)

The second edition gives us the opportunity to present this new version of our book: It contains three additional chapters, substantial revisions and new proofs in several others, as well as minor amendments and improvements, many of them based on the suggestions we received. It also misses one of the old chapters, about the “problem of the thirteen spheres,” whose proof turned out to need details that we couldn't complete in a way that would make it brief and elegant.

Thanks to all the readers who wrote and thus helped us — among them Stephan Brandt, Christian Elsholtz, Jürgen Elstroth, Daniel Grieser, Roger Heath-Brown, Lee L. Keener, Hanfried Lenz, Nicolas Puech, John Scholes, Bernulf Weißbach, and *many* others. Thanks again for help and support to Ruth Allewelt and Karl-Friedrich Koch at Springer Heidelberg, to Christoph Eyrich and Torsten Heldmann in Berlin, and to Karl H. Hofmann for some superb new drawings.

Berlin, September 2000

Martin Aigner · Günter M. Ziegler

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