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Local Algebra

Translated from the French by CheeWhye Chin

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Preface

The present book is an English translation of

Algèbre Locale — Multiplicités

published by Springer-Verlag as no. 11 of the Lecture Notes series.

The original text was based on a set of lectures, given at the Collège de France in 1957-1958, and written up by Pierre Gabriel. Its aim was to give a short account of Commutative Algebra, with emphasis on the following topics:

a) *Modules* (as opposed to *Rings*, which were thought to be the only subject of Commutative Algebra, before the emergence of sheaf theory in the 1950s);
b) *Homological methods*, à la Cartan-Eilenberg;
c) *Intersection multiplicities*, viewed as Euler-Poincaré characteristics.

The English translation, done with great care by CheeWhye Chin, differs from the original in the following aspects:
– The terminology has been brought up to date (e.g. "cohomological dimension" has been replaced by the now customary "depth").
– I have rewritten a few proofs and clarified (or so I hope) a few more.
– A section on graded algebras has been added (App. III to Chap. IV).
– New references have been given, especially to other books on Commutative Algebra: Bourbaki (whose Chap. X has now appeared, after a 40-year wait), Eisenbud, Matsumura, Roberts, ...

I hope that these changes will make the text easier to read, without changing its informal "Lecture Notes" character.

J-P. Serre,
Princeton, Fall 1999

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Introduction

The intersection multiplicities of algebraic geometry are equal to some "Euler-Poincaré characteristics" constructed by means of the Tor functor of Cartan-Eilenberg. The main purpose of this course is to prove this result, and to apply it to the fundamental formulae of intersection theory.

It is necessary to first recall some basic results of local algebra: primary decomposition, Cohen-Seidenberg theorems, normalization of polynomial rings, Krull dimension, characteristic polynomials (in the sense of Hilbert-Samuel).

Homology comes next, when we consider the multiplicity $e_{\mathfrak{q}}(E, r)$ of an ideal of definition $\mathfrak{q} = (x_1, \dots, x_r)$ of a local noetherian ring A with respect to a finitely generated A-module E. This multiplicity is defined as the coefficient of $n^r/r!$ in the polynomial-like function $n \mapsto \ell_A(E/\mathfrak{q}^n E)$ [here $\ell_A(F)$ is the length of an A-module F]. We prove in this case the following formula, which plays an essential role in the sequel:

$$e_{\mathfrak{q}}(E, r) = \sum_{i=0}^{r} (-1)^i \, \ell_A(H_i(\mathbf{x}, E)) \qquad (*)$$

where the $H_i(\mathbf{x}, E)$ denotes the homology modules of the Koszul complex constructed on E by means of $\mathbf{x} = (x_1, \dots, x_r)$.

Moreover this complex can be used in other problems of local algebra, for example for the study of the depth of modules over a local ring and of the Cohen-Macaulay modules (those whose Krull dimension coincides with their depth), and also for showing that regular local rings are the only local rings whose homological dimension is finite.

Once formula $(*)$ is proved, one may study the Euler-Poincaré characteristic constructed by means of Tor. When one translates the geometric situation of intersections into the language of local algebra, one obtains a regular local ring A, of dimension n, and two finitely generated A-modules E and F over A, whose tensor product is of finite length over A (this means that the varieties corresponding to E and F intersect only at the given point). One is then led to conjecture the following statements:

(i) $\dim(E) + \dim(F) \leq n$ ("dimension formula").

(ii) $\chi_A(E,F) = \sum_{i=0}^{n}(-1)^i \ell_A(\mathrm{Tor}_i^A(E,F))$ is ≥ 0.

(iii) $\chi_A(E,F) = 0$ if and only if the inequality in (i) is strict.

Formula $(*)$ shows that the statements (i), (ii) and (iii) are true if $F = A/(x_1, \ldots, x_r)$, with $\dim(F) = n - r$. Thanks to a process, using completed tensor products, which is the algebraic analogue of "reduction to the diagonal", one can show that they are true when A has the same characteristic as its residue field, or when A is unramified. To go beyond that, one can use the structure theorems of complete local rings to prove (i) in the most general case. On the other hand, I have not succeeded in proving (ii) and (iii) without making assumptions about A, nor to give counter-examples. It seems that it is necessary to approach the question from a different angle, for example by directly defining (by a suitable asymptotic process) an integer ≥ 0 which one would subsequently show to be equal to $\chi_A(E,F)$.

Fortunately, the case of equal characteristic is sufficient for the applications to algebraic geometry (and also to analytic geometry). More specifically, let X be a non-singular variety, let V and W be two irreducible subvarieties of X, and suppose that $C = V \cap W$ is an irreducible subvariety of X, with:

$$\dim X + \dim C = \dim V + \dim W \qquad \text{("proper" intersection).}$$

Let A, A_V, A_W be the local rings of X, V and W at C. If

$$i(V \cdot W, C; X)$$

denotes the multiplicity of the intersection of V and W at C (in the sense of Weil, Chevalley, Samuel), we have the formula:

$$i(V \cdot W, C; X) = \chi_A(A_V, A_W). \qquad (**)$$

This formula is proved by reduction to the diagonal, and the use of $(*)$. In fact, it is convenient to take $(**)$ as the definition of multiplicities. The properties of these multiplicities are then obtained in a natural way: commutativity follows from that for Tor; associativity follows from the two spectral sequences which expresses the associativity of Tor; the projection formula follows from the two spectral sequences connecting the direct images of a coherent sheave and Tor (these latter spectral sequences have other interesting applications, but they are not explored in the present course). In each case, one uses the well-known fact that Euler-Poincaré characteristics remain constant through a spectral sequence.

When one defines intersection multiplicities by means of the Tor-formula above, one is led to extend the theory beyond the strictly "non-singular" framework of Weil and Chevalley. For example, if $f : X \to Y$

is a morphism of a variety X into a non-singular variety Y, one can associate, to two cycles x and y of X and Y, a "product" $x \cdot_f y$ which corresponds to $x \cap f^{-1}(y)$ (of course, this product is only defined under certain dimension conditions). When f is the identity map, one recovers the standard product. The commutativity, associativity and projection formulae can be stated and proved for this new product.