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Galerkin
Finite Element Methods for Parabolic Problems

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Preface

My purpose in this monograph is to present an essentially self-contained account of the mathematical theory of Galerkin finite element methods as applied to parabolic partial differential equations. The emphases and selection of topics reflects my own involvement in the field over the past 25 years, and my ambition has been to stress ideas and methods of analysis rather than to describe the most general and farreaching results possible. Since the formulation and analysis of Galerkin finite element methods for parabolic problems are generally based on ideas and results from the corresponding theory for stationary elliptic problems, such material is often included in the presentation.

The basis of this work is my earlier text entitled *Galerkin Finite Element Methods for Parabolic Problems,* Springer Lecture Notes in Mathematics, No. 1054, from 1984. This has been out of print for several years, and I have felt a need and been encouraged by colleagues and friends to publish an updated version. In doing so I have included most of the contents of the 14 chapters of the earlier work in an updated and revised form, and added four new chapters, on semigroup methods, on multistep schemes, on incomplete iterative solution of the linear algebraic systems at the time levels, and on semilinear equations. The old chapters on fully discrete methods have been reworked by first treating the time discretization of an abstract differential equation in a Hilbert space setting, and the chapter on the discontinuous Galerkin method has been completely rewritten.

The following is an outline of the contents of the book:

In the introductory Chapter 1 we begin with a review of standard material on the finite element method for Dirichlet's problem for Poisson's equation in a bounded domain, and consider then the simplest Galerkin finite element methods for the corresponding initial-boundary value problem for the linear heat equation. The discrete methods are based on associated weak, or variational, formulations of the problems and employ first piecewise linear and then more general approximating functions which vanish on the boundary of the domain. For these model problems we demonstrate the basic error estimates in energy and mean square norms, in the parabolic case first for the semidiscrete problem resulting from discretization in the spatial variables only, and then also for the most commonly used fully discrete schemes obtained by discretization in both space and time, such as the backward Euler and Crank-Nicolson methods.

In the following five chapters we study several extensions and generalizations of the results obtained in the introduction in the case of the spatially semidiscrete approximation, and show error estimates in a variety of norms. First, in Chapter 2, we formulate the semidiscrete problem in terms of a more general approximate solution operator for the elliptic problem in a manner which does not require the approximating functions to satisfy the homogeneous boundary conditions. As an example of such a method we discuss a method of Nitsche based on a nonstandard weak formulation. In Chapter 3 more precise results are shown in the case of the homogeneous heat equation. These results are expressed in terms of certain function spaces $\dot{H}^s(\Omega)$ which are characterized by both smoothness and boundary behavior of its elements, and which will be used repeatedly in the rest of the book. We also demonstrate that the smoothing property for positive time of the solution operator of the initial value problem has an analogue in the semidiscrete situation, and use this to show that the finite element solution converges to full order even when the initial data are nonsmooth. The results of Chapters 2 and 3 are extended to more general linear parabolic equations in Chapter 4. Chapter 5 is devoted to the derivation of stability and error bounds with respect to the maximum-norm for our plane model problem, and in Chapter 6 negative norm error estimates of higher order are derived, together with related results concerning superconvergence.

In the next six chapters we consider fully discrete methods obtained by discretization in time of the spatially semidiscrete problem. First, in Chapter 7, we study the homogeneous heat equation and give analogues of our previous results both for smooth and for nonsmooth data. The methods used for time discretization are of one-step type and rely on rational approximations of the exponential, allowing the standard Euler and Crank-Nicolson procedures as special cases. Our approach here is to first discretize a parabolic equation in an abstract Hilbert space framework with respect to time, and then to apply the results obtained to the spatially semidiscrete problem. The analysis uses eigenfunction expansions related to the elliptic operator occurring in the parabolic equation, which we assume positive definite. In Chapter 8 we generalize the above abstract considerations to a Banach space setting and allow a more general parabolic equation, which we now analyze using the Dunford-Taylor spectral representation. The time discretization is interpreted as a rational approximation of the semigroup generated by the elliptic operator, i.e., the solution operator of the initial-value problem for the homogeneous equation. Application to maximum-norm estimates is discussed. In Chapter 9 we study fully discrete one-step methods for the inhomogeneous heat equation in which the forcing term is evaluated at a fixed finite number of points per time stepping interval. In Chapter 10 we apply Galerkin's method also for the time discretization and seek discrete solutions as piecewise polynomials in the time variable which may be discontinuous at the now not necessarily equidistant nodes. In this *discontinuous Galerkin* procedure the forcing term enters in integrated form rather than at a finite number of points. In Chapter 11 we consider multistep backward difference methods. We first study such methods with constant time steps of order at most 6, and show stability as well as smooth and nonsmooth data error estimates, and then discuss the second order backward difference method with variable time steps. In Chapter 12 we study the incomplete iterative solution of the finite dimensional linear systems of algebraic equations which need to be solved at each level of the time stepping procedure, and exemplify by the use of a V-cycle multigrid algorithm.

The next two chapters are devoted to nonlinear problems. In Chapter 13 we discuss the application of the standard Galerkin method to a model nonlinear parabolic equation. We show error estimates for the spatially semidiscrete problem as well as the fully discrete backward Euler and Crank-Nicolson methods, using piecewise linear finite elements, and then pay special attention to the formulation and analysis of time stepping procedures based on these, which are linear in the unknown functions. In Chapter 14 we derive various results in the case of semilinear equations, in particular concerning the extension of the analysis for nonsmooth initial data from the case of linear homogenous equations.

In the last four chapters we consider various modifications of the standard Galerkin finite element method. In Chapter 15 we analyze the so called lumped mass method for which in certain cases a maximum-principle is valid. In Chapter 16 we discuss the H^1 and H^{-1} methods. In the first of these, the Galerkin method is based on a weak formulation with respect to an inner product in H^1 and for the second, the method uses trial and test functions from different finite dimensional spaces. In Chapter 17, the approximation scheme is based on a mixed formulation of the initial boundary value problem in which the solution and its gradient are sought independently in different spaces. In the final Chapter 18 we consider a singular problem obtained by introducing polar coordinates in a spherically symmetric problem in a ball in $\mathbf{R}^3$ and discuss Galerkin methods based on two different weak formulations defined by two different inner products.

References to the literature where the reader may find more complete treatments of the different topics, and some historical comments, are given at the end of each chapter.

A desirable mathematical background for reading the text includes standard basic partial differential equations and functional analysis, including Sobolev spaces; for the convenience of the reader we often give references to the literature concerning such matters.

The work presented, first in the Lecture Notes and now in this monograph, has grown from courses, lecture series, summer-schools, and written material that I have been involved in over a long period of time. I wish to thank my

students and colleagues in these various contexts for the inspiration and support they have provided, and for the help they have given me as discussion partners and critics. As regards this new version of my work I particularly address my thanks to Georgios Akrivis, Stig Larsson, and Per-Gunnar Martinsson, who have read the manuscript in various degrees of detail and are responsible for many improvements. I also want to express my special gratitude to Yumi Karlsson who typed a first version of the text from the old lecture notes, and to Gunnar Ekolin who generously furnished me with expert help with the intricacies of TeX.

Göteborg, July 1997

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