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Sheaves on Manifolds

With a Short History

«Les débuts de la théorie des faisceaux»

By Christian Houzel



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Preface

For a long time after its introduction by Leray, sheaf theory was mainly applied to the theory of functions of several complex variables or to algebraic geometry, until it became a basic tool for almost all mathematicians, and cohomology a natural language for many people.

However, while there exists an extensive literature dealing with cohomology of sheaves (e.g. the famous book by Godement) or even with derived functors, there are in fact very few books developing sheaf theory within the beautiful framework of derived categories although its necessity is becoming more and more evident. Most of the constructions of the theory take on their full strength in this context, or even, do not make sense outside of it. This is particularly evident for the Poincaré-Verdier duality, which appeared in the sixties, as well as for the Sato microlocalization, introduced in 1969, which is only beginning to be fully understood.

Since the seventies, other fundamental ideas have emerged and sheaf theory (on manifolds) naturally includes the “microlocal” point of view. Our aim is to present here a self-contained work, starting from the beginning (derived categories and sheaves), dealing in detail with the main features of the theory, such as duality, Fourier transformation, specialization and microlocalization, microsupport and contact transformations, and also to give two main applications. The first of these deals with real analytic geometry, and includes the concepts of constructible sheaves, subanalytic cycles, Euler-Poincaré indices, Lefschetz formula, perverse sheaves, etc. The second one is the theory of linear partial differential equations, including D -modules, microfunctions, elliptic and micro-hyperbolic systems, and complex quantized contact transformations.

With this book we hope to illustrate the deep links that tie together branches of mathematics at first glance seemingly disconnected, such as for example here, algebraic topology and linear partial differential equations. At the same time, we want to emphasize the essentially geometrical nature of the problems encountered (most obvious in the involutivity theorem for sheaves), and to show how efficient the algebraic tools introduced by Grothendieck are in solving them, even for an analyst.

Of course, many important applications of the theory are just touched upon, such as for instance the theory of microdifferential systems (complete monographs on the topic are however available now), others are simply omitted, such as representation theory and equivariant sheaf theory.

Finally, we want to express our thanks to C. Houzel who agreed to write a short history of sheaf theory, to L. Illusie who helped us when preparing the “Historical Notes”, to those who went through various parts of the book and made constructive comments, especially E. Andronikof, A. Arabia, J-M. Delort, E. Leichtnam and J-P. Schneiders, and also to Catherine Simon at Paris-Nord University and the secretarial staff of the RIMS at Kyoto, who had the patience to type the manuscripts.

May 1990

M. Kashiwara and P. Schapira

Table of contents

Introduction	1
A Short History: Les débuts de la théorie des faisceaux by Christian Houzel	7
I. Homological algebra	23
Summary	23
1.1. Categories and functors	23
1.2. Abelian categories	26
1.3. Categories of complexes	30
1.4. Mapping cones	34
1.5. Triangulated categories	38
1.6. Localization of categories	41
1.7. Derived categories	45
1.8. Derived functors	50
1.9. Double complexes	54
1.10. Bifunctors	56
1.11. Ind-objects and pro-objects	61
1.12. The Mittag-Leffler condition	64
Exercises to Chapter I	69
Notes	81
II. Sheaves	83
Summary	83
2.1. Presheaves	83
2.2. Sheaves	85
2.3. Operations on sheaves	90
2.4. Injective, flabby and flat sheaves	98
2.5. Sheaves on locally compact spaces	102
2.6. Cohomology of sheaves	109
2.7. Some vanishing theorems	116
2.8. Cohomology of coverings	123
2.9. Examples of sheaves on real and complex manifolds	125

Exercises to Chapter II	131
Notes	138
III. Poincaré-Verdier duality and Fourier-Sato transformation	139
Summary	139
3.1. Poincaré-Verdier duality	140
3.2. Vanishing theorems on manifolds	149
3.3. Orientation and duality	151
3.4. Cohomologically constructible sheaves	158
3.5. γ -topology	161
3.6. Kernels	164
3.7. Fourier-Sato transformation	167
Exercises to Chapter III	178
Notes	184
IV. Specialization and microlocalization	185
Summary	185
4.1. Normal deformation and normal cones	185
4.2. Specialization	190
4.3. Microlocalization	198
4.4. The functor μ_{hom}	201
Exercises to Chapter IV	214
Notes	215
V. Micro-support of sheaves	217
Summary	217
5.1. Equivalent definitions of the micro-support	218
5.2. Propagation	222
5.3. Examples: micro-supports associated with locally closed subsets	226
5.4. Functorial properties of the micro-support	229
5.5. Micro-support of conic sheaves	241
Exercises to Chapter V	245
Notes	247
VI. Micro-support and microlocalization	249
Summary	249
6.1. The category $\mathbf{D}^b(X; \Omega)$	250
6.2. Normal cones in cotangent bundles	258
6.3. Direct images	263
6.4. Microlocalization	268
6.5. Involutivity and propagation	271

6.6. Sheaves in a neighborhood of an involutive manifold	274
6.7. Microlocalization and inverse images	275
Exercises to Chapter VI	279
Notes	281
VII. Contact transformations and pure sheaves	283
Summary	283
7.1. Microlocal kernels	284
7.2. Contact transformations for sheaves	289
7.3. Microlocal composition of kernels	293
7.4. Integral transformations for sheaves associated with submanifolds ..	298
7.5. Pure sheaves	309
Exercises to Chapter VII	318
Notes	318
VIII. Constructible sheaves	320
Summary	320
8.1. Constructible sheaves on a simplicial complex	321
8.2. Subanalytic sets	327
8.3. Subanalytic isotropic sets and μ -stratifications	328
8.4. \mathbb{R} -constructible sheaves	338
8.5. \mathbb{C} -constructible sheaves	344
8.6. Nearby-cycle functor and vanishing-cycle functor	350
Exercises to Chapter VIII	356
Notes	358
IX. Characteristic cycles	360
Summary	360
9.1. Index formula	361
9.2. Subanalytic chains and subanalytic cycles	366
9.3. Lagrangian cycles	373
9.4. Characteristic cycles	377
9.5. Microlocal index formulas	384
9.6. Lefschetz fixed point formula	389
9.7. Constructible functions and Lagrangian cycles	398
Exercises to Chapter IX	406
Notes	409
X. Perverse sheaves	411
Summary	411
10.1. t -structures	411
10.2. Perverse sheaves on real manifolds	419

10.3. Perverse sheaves on complex manifolds	426
Exercises to Chapter X	438
Notes	440
XI. Applications to \mathcal{O}-modules and \mathcal{D}-modules	441
Summary	441
11.1. The sheaf \mathcal{O}_X	442
11.2. \mathcal{D}_X -modules	445
11.3. Holomorphic solutions of \mathcal{D}_X -modules	453
11.4. Microlocal study of \mathcal{O}_X	459
11.5. Microfunctions	466
Exercises to Chapter XI	471
Notes	474
Appendix: Symplectic geometry	477
Summary	477
A.1. Symplectic vector spaces	477
A.2. Homogeneous symplectic manifolds	481
A.3. Inertia index	486
Exercises to the Appendix	493
Notes	495
Bibliography	496
List of notations and conventions	502
Index	509