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Introductory Optimization Dynamics

Optimal Control with Economics and
Management Science Applications

With 85 Figures

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PREFACE

Optimal Control theory has been increasingly used in Economics and Management Science in the last fifteen years or so. It is now commonplace, even at textbook level. It has been applied to a great many areas of Economics and Management Science, such as Optimal Growth, Optimal Population, Pollution control, Natural Resources, Bioeconomics, Education, International Trade, Monopoly, Oligopoly and Duopoly, Urban and Regional Economics, Arms Race control, Business Finance, Inventory Planning, Marketing, Maintenance and Replacement policy and many others. It is a powerful tool of dynamic optimization.

There is no doubt social sciences students should be familiar with this tool, if not for their own research, at least for reading the literature. These Lecture Notes attempt to provide a plain exposition of Optimal Control Theory, with a number of economic examples and applications designed mainly to illustrate the various techniques and point out the wide range of possible applications rather than to treat exhaustively any area of economic theory or policy. Chapters 2,3 and 4 are devoted to the Calculus of Variations, Chapter 5 develops Optimal Control theory from the Variational approach, Chapter 6 deals with the problems of constrained state and control variables, Chapter 7, with Linear Control models and Chapter 8, with stabilization models. Discrete systems are discussed in Chapter 9 and Sensitivity analysis in Chapter 10. Chapter 11 presents a wide range of Economics and Management Science applications. Only deterministic Control Theory will be dealt with: Stochastic Control, Differential Games and other related topics are beyond the scope of these introductory Lecture Notes.

My motivation to write this introductory text originates from a realisation, some six years ago, of the Economics, Business and other Social Sciences students' need of a simple text on Optimal Control Theory tailored to their own requirements and illustrated with familiar examples on the one hand and the scarcity at the time of such books, on the other. Students faced two extreme choices: they could either acquire a superficial understanding of the Theory by reading a chapter or two in Mathematical Economics textbooks, or wade their way through Mathematics and Engineering texts at the risk of getting lost by the degree of abstraction on the one hand and the unfamiliar Engineering examples on the other. Instructors had to write their own texts. These Lecture Notes cover a major part of a half course given at the University of Calgary to junior graduate and senior undergraduate Economics students with no more than a general knowledge of Linear Algebra and Calculus, including Differential and Difference Equations. The Mathematical Appendix is intended to remedy any further deficiencies students might have in Differential and Difference Equations.

No originality is normally claimed for Lecture Notes. These are no exception. The author owes an immense debt to the literature (listed in the References) which is so rich and varied that it is almost impossible to acknowledge all sources individually. No attempts will be made in this direction. The students who have taken the course have undoubtedly contributed much to the improvement of the exposition and reduction of fallacies. My colleagues have also made their contribution through intellectual conversations. Professors M.C. Kemp, R. Pindyck and R. Dobell have also made some suggestions which are much appreciated.

My thanks are also extended to the Department of A & M Economics of the University of Southampton where I spent my sabbatical leave in 1980-1981 for the provision of an ideal environment and research facilities needed to bring the first draft of this manuscript to completion. Mary Blount is to be commended for her exemplary patience and technical competence in the typing of most the manuscript. Finally, this publication has been made possible, in part, by a grant from the Endowment Fund of the University of Calgary, for which I am grateful. Needless to say that the remaining errors and deficiencies are mine alone.

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