

To the Memory of our father  
Otto Nevanlinna

# Die Grundlehren der mathematischen Wissenschaften

in Einzeldarstellungen  
mit besonderer Berücksichtigung  
der Anwendungsgebiete

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# Absolute Analysis

Translated from the German  
by Phillip Emig

With 5 Figures



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## Foreword

The first edition of this book, published in German, came into being as the result of lectures which the authors held over a period of several years since 1953 at the Universities of Helsinki and Zürich. The Introduction, which follows, provides information on what motivated our presentation of an absolute, coordinate- and dimension-free infinitesimal calculus.

Little previous knowledge is presumed of the reader. It can be recommended to students familiar with the usual structure, based on coordinates, of the elements of analytic geometry, differential and integral calculus and of the theory of differential equations.

We are indebted to H. Keller, T. Klemola, T. Nieminen, Ph. Tondeur and K. I. Virtanen, who read our presentation in our first manuscript, for important critical remarks.

The present new English edition deviates at several points from the first edition (cf. Introduction). Professor I. S. Louhivaara has from the beginning to the end taken part in the production of the new edition and has advanced our work by suggestions on both content and form. For his important support we wish to express our hearty thanks. We are indebted also to W. Greub and to H. Hahti for various valuable remarks.

Our manuscript for this new edition has been translated into English by Doctor P. Emig. We express to him our gratitude for his careful interest and skillful attention during this work.

Our thanks are also due to Professor F. K. Schmidt and to Springer-Verlag for including this new edition in the „Grundlehren der mathematischen Wissenschaften“ series.

Helsinki, May 1973

THE AUTHORS

## Table of Contents

Introduction . . . . .	1
I. Linear Algebra . . . . .	4
§ 1. The Linear Space with Real Multiplier Domain . . . . .	4
§ 2. Finite Dimensional Linear Spaces . . . . .	10
§ 3. Linear Mappings . . . . .	15
§ 4. Bilinear and Quadratic Functions . . . . .	26
§ 5. Multilinear Functions . . . . .	41
§ 6. Metrization of Affine Spaces. . . . .	56
II. Differential Calculus . . . . .	74
§ 1. Derivatives and Differentials . . . . .	74
§ 2. Taylor's Formula . . . . .	90
§ 3. Partial Differentiation . . . . .	95
§ 4. Implicit Functions. . . . .	99
III. Integral Calculus . . . . .	118
§ 1. The Affine Integral . . . . .	118
§ 2. Theorem of Stokes. . . . .	124
§ 3. Applications of Stokes's Theorem . . . . .	140
IV. Differential Equations . . . . .	148
§ 1. Normal Systems . . . . .	148
§ 2. The General Differential Equation of First Order . . . . .	158
§ 3. The Linear Differential Equation of Order One . . . . .	163
V. Theory of Curves and Surfaces . . . . .	181
§ 1. Regular Curves and Surfaces . . . . .	181
§ 2. Curve Theory . . . . .	185
§ 3. Surface Theory . . . . .	193
§ 4. Vectors and Tensors . . . . .	202
§ 5. Integration of the Derivative Formulas. . . . .	221
§ 6. Theorema Egregium . . . . .	228
§ 7. Parallel Translation . . . . .	232
§ 8. The Gauss-Bonnet Theorem. . . . .	240
VI. Riemannian Geometry . . . . .	247
§ 1. Affine Differential Geometry . . . . .	248
§ 2. Riemannian Geometry . . . . .	257
Bibliography . . . . .	261
Index . . . . .	264