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# Compact Complex Surfaces



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## Preface

Par une belle matinée du mois de mai,  
une élégante amazone parcourait, sur  
une superbe jument alezane, les  
allées fleuries du Bois de Boulogne.

(A. Camus, *La Peste*)

Early versions of parts of this work date back to the mid-sixties, when the third author started to write a book on surfaces. But for several reasons, in particular the appearance of Šafarevič's book, he postponed the project. It was revived about ten years later, when all three authors were in Leiden.

It is impossible to cover in one book the vast and rapidly developing theory of surfaces. Choices have to be made, with respect to content as well as to presentation. We have chosen for a complex-analytic point of view; this distinguishes our text from most of the existing treatments. Relations with the case of characteristic  $p$  are not discussed.

We hope to have succeeded in writing a readable book; a book that can be used by non-specialists. The specialist will find very little that is new to him anyhow.

As to acknowledgements, the authors certainly have to mention the Koninklijke Shellprijs, awarded to the third author in 1964. The numerous contacts with colleagues from other countries made possible by that award have had a very favourable influence on this book.

Our thanks are furthermore due to G. Angermüller, G. Barthel, G. Fischer, G. van der Geer, N. Hitchin, D. Husemoller, M. Reid, T.A. Springer, D. Zagier and S. Zucker. Each of them has read some part of the manuscript and has made valuable suggestions.

Editor and printer have done an excellent job, and the Springer-Verlag has been very generous in fulfilling all of our last-minute wishes.

We are also indebted to Mrs. W.M. Van de Ven who not only typed the better part of the book, but also helped in preparing it for the printer, and to Mrs. H. Dohrman who carefully typed many pages.

Finally the authors want to thank their wives for all their patience and endurance.

Erlangen/Leiden, February 1984

W. Barth  
C. Peters  
A. Van de Ven

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