IV. Structure of Abelian Categories

An abelian category C with coproducts is cocomplete 6.3, and is called an $AB3$ category. The sum of any family of subobjects of an object exist, and if, in addition, sum distributes over intersection in the sense that

$$\sum_{i\in I} A_i \cap B = \sum_{i\in I} (A_i \cap B)$$

where $\{A_i\}_{i\in I}$ is any directed family of subobjects of any object A of C and B is any subobject of A, then C is said to be an $AB5$ category (Grothendieck [57]). (An $AB5$ category in which sum distributes over arbitrary intersections is called an $AB6$ category. See Chapter 15 Exercises.) An $AB3$ category is $AB5$ if and only if it has exact direct limits 14.6, or equivalently, the direct limit of any directed family of subobjects is a subobject (and in this case the direct limit is the sum).

A Grothendieck category is an $AB5$ category with a generator. In such a category, every object has an injective hull 14.17. This is needed to prove the main result of Chapter 14, the (Freyd, Lubkin) embedding theorem 14.21, which states that every small abelian category C has an exact, covariant embedding $C \rightsquigarrow$ mod-$\mathbb{Z}$. The (Mitchell) full embedding theorem 15.30 states that a small abelian category C has a full exact embedding $C \rightsquigarrow$ mod-A, for a suitable ring A. (This also provides an exact embedding $C \rightsquigarrow$ mod-$\mathbb{Z}$.)

Quotient categories are introduced in Chapter 15. A full subcategory $\mathscr{S}$ of an abelian category $\mathscr{A}$ is called a **Serre class** if for every exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{A}$ it is true that $Y \in \mathscr{S} \Leftrightarrow X \,\&\, Z \in \mathscr{S}$. Then, there is a category $\mathscr{A}/\mathscr{S}$ on the objects of $\mathscr{A}$, and morphisms are defined in $\mathscr{A}/\mathscr{S}$ by:

$$\mathrm{Mor}_{\mathscr{A}/\mathscr{S}}(X, Y) = \varinjlim_{Y', X/X' \in \mathscr{S}} \mathrm{Mor}_{\mathscr{A}}(X', Y/Y')$$

The category $\mathscr{A}/\mathscr{S}$ is called a **quotient category**, and is abelian 15.7. The Serre subcategory $\mathscr{S}$ is a **localizing** subcategory provided that the canonical functor $T: \mathscr{A} \rightsquigarrow \mathscr{A}/\mathscr{S}$ has a right adjoint $S: \mathscr{A}/\mathscr{S} \rightsquigarrow \mathscr{A}$ (called the **section** functor), and the composition $\mathscr{A} \rightsquigarrow \mathscr{A}$ is called the **localizing** functor. If $\mathscr{A}$ is Grothendieck, a necessary and sufficient condition for this is that $\mathscr{S}$ should be a subcategory with arbitrary coproducts, or equivalently, direct limits taken in $\mathscr{A}$ of directed families of subobjects of $\mathscr{S}$ should belong to $\mathscr{S}$. In this case, for each object M of $\mathscr{A}$, there is an exact sequence

$$0 \to X \to M \xrightarrow{u} N \to Y \to 0$$

where X, $Y \in \mathscr{S}$, and N is $\mathscr{S}$-closed in the sense 15.15. If C denotes the full subcategory of $\mathscr{S}$-closed objects, then the localizing functor induces an equivalence

$$\psi(N) : N \to STN,$$

and there is an equivalence of categories $\mathscr{A}/\mathscr{S} \approx C$. The morphism $u : M \to N$ is determined uniquely up to an equivalence of $\mathscr{A}$, and is called the $\mathscr{S}$-envelope of M, denoted $u_{\mathscr{S}} : M \to M_{\mathscr{S}}$. If $\mathscr{A} = \text{mod-}A$, for some ring A, then $A_{\mathscr{S}}$ is a ring, and in classical situations 16.8 when the localizing functor is exact, this functor is naturally isomorphic to $\otimes_A A_{\mathscr{S}}$, and there is an equivalence of categories

$$\text{mod-}A/\mathscr{S} \approx \text{mod-}A_{\mathscr{S}}.$$

This situation occurs for the localization theorem 16.15 generalizing to Grothendieck categories localization theorems for commutative rings (Chapters 27 and 28). This method also applies to partial right quotient rings 16.9 generalizing the right quotient rings of Chapters 9 and 10, and to the Johnson maximal right quotient rings of right neat rings 16.12. (Cf. also Chapter 19.)

A classical aspect of module theory is the relating to a ring A intrinsically some extrinsic property of mod-A. Thus, 8.12 every module is semisimple if and only if A is semisimple or 8.21,

$$\text{r.gl. dim } A = \sup \{\dim I \mid I \subseteq R\} + 1$$

when A is not semisimple; or the characterisation 11.24 of rings with every module flat, as regular rings in which every principal right ideal is generated by an idempotent. These examples can be multiplied at will, for in a sense, this is ring (and module) theory.

In this vein, a natural question raised by localizing subcategories of mod-A is the intrinsic relation to mod-A. This is solved by Gabriel's theorem 16.3 which establishes a mapping

$$C \mapsto \mathscr{F}(C) = \{I \subseteq A \mid A/I \in \text{obj } C\}$$

of the class of localizing subcategories of mod-A to the class consisting of sets of all right ideals of A. The set $\mathscr{F}(C)$ is idempotent and topologizing

(= idemtop) in the sense of 16.3, and conversely, any idemtop set F of right ideals of A corresponds to the localizing subcategory $\mathscr{S}(F)$ of mod-A consisting of all modules M such that for every $x \in M$ the right ideal $\operatorname{ann}_A x \in F$. Moreover, as stated above for Grothendieck categories the quotient category mod-$A/\mathscr{S}$ is equivalent to the full subcategory C of mod-A consisting of $\mathscr{S}$-closed modules, or in the terminology of torsion theories taken up in Chapter 16, the subcategory consisting of torsionfree divisible modules. The same correspondence can be used to characterize Serre classes, and classes of closed full subcategories of mod-A 16.2. In these cases, then the $\mathscr{S}(F)$-envelope of a module M is denoted M_F instead of $M_{\mathscr{S}(F)}$, and in particular, A_F denotes $A_{\mathscr{S}(F)}$.

Chapter 15 contains the Popesco-Gabriel theorem 15.26 which characterizes Grothendieck categories as quotient categories of module type categories.

Chapter 16 concludes with a theorem 16.20 of E. A. and C. L. Walker stating that mod-A is determined in a very strong sense by the endomorphism ring of any proper generator. This generalizes the known theorems for principal right ideal domains, and quasi-Frobenius rings. (Cf. Chapter 24.)

Chapter 14

Grothendieck Categories

Grothendieck [57] introduced the notation for abelian categories which follows:

$AB3$. A coproductive, hence cocomplete, abelian category.

$AB4$. An $AB3$ category in which every coproduct $\coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ of monics $\{A_i \to B_i\}_{i\in I}$ is monic.

$AB5$. An $AB3$ category such that for any directed family $\{A_i\}_{i\in I}$ of subobjects of any object X, and any subobject B,

$$\Big(\sum_{i\in I} A_i\Big) \cap B = \sum_{i\in I} (A_i \cap B).$$

ABX^*, the dual of ABX, $X = 3, 4, 5$.

For any totally ordered family $\{A_i\}_{i\in I}$ of subobjects of any object X, the sum is called **union**, and is denoted by $\bigcup_{i\in I} A_i$. If the family is finite, and consists of $A_1 \leq \cdots \leq A_n$, then $\bigcup_{i\in I} A_i = A_n$.