III. Tensor Algebra

If M is a (B, A)-bimodule, then $\mathrm{Hom}_A(M, X)$ is canonically a right B-module, for any object X of mod-A, and the resulting functor

$$\mathrm{Hom}_A(M,\ )\begin{cases} \text{mod-}A \rightsquigarrow \text{mod-}B \\ \quad X \mapsto \mathrm{Hom}_A(M, X) \end{cases}$$

is also denoted $\mathrm{Hom}_A(M,\ )$, or simply h^M. (Dually for $\mathrm{Hom}_A(\ , M) = h_M$.) This functor is left continuous in the sense that it preserves products and kernels 5.44, and the left adjoint

$$\underset{B}{\otimes} M \begin{cases} \text{mod-}B \rightsquigarrow \text{mod-}A \\ \quad Y \mapsto Y \underset{B}{\otimes} M \end{cases}$$

is right continuous, that is, preserves coproducts and cokernels. (Cf. Adjoint associativity 11.4.) The freedom of the tensor product of free modules is a consequence 11.10.

A module M is flat in mod-B if and only if $\underset{B}{\otimes} M : \text{mod-}B \rightsquigarrow \text{mod-}\mathbb{Z}$ is exact. The fundamental properties of flat modules are taken up in Chapter 11. Thus, a module is flat if and only if M is the direct limit of projective modules (see 11.32 and 6.24). Hence, every projective module is flat 11.22. Furthermore, every finitely related flat module is projective. Over a Noetherian ring, this implies that every finitely generated flat module is projective 11.31. Regular rings are characterized 11.24 by the property that every right module is flat. (This property is right-left symmetric.) As is shown in Chapter 19, the injective hull R of any right neat ring R (that is, a ring with zero right singular ideal) is a right self-injective regular ring, and more generally, the endomorphism ring of any quasinjective non-singular right R-module M is regular and right selfinjective. (Any right full linear ring is such an example.)

The main content of Chapter 12 is an exposition of the Morita theorems using tensor products in a form suitable for the descriptions of the structure of the Picard group coming at the end of the chapter, and the Brauer group in Chapter 32. Thus, $\mathrm{Pic}_k A$ is the group of isomorphism classes (P) of invertible (A, A)-bimodules under the operation $(P)(Q) = \left(P \underset{k}{\otimes} Q\right)$, where A is any algebra over a commutative ring k.

There is an isomorphism 12.15

$$\begin{cases} \text{Pic}_k A \to \text{Pic}\,(A\text{-mod}) \\ \quad (P) \mapsto P \underset{A}{\otimes} \\ \quad TA \leftarrow\!\shortmid\, T \end{cases}$$

where Pic (mod-A) is the group of k-linear autoequivalences of mod-A. If A is commutative, then there is an isomorphism $\text{Aut}_k A \approx \text{Pic}_k A / \text{Pic}_A A$ of the group of k-algebra automorphisms 12.18.

Chapter 13 is an exposition of some properties of finite dimensional algebras over a field k. These include the Wedderburn-Artin theorem 13.5 for algebraically closed k, the nilpotence of algebras with nil basis 13.28, and the theorem of Maschke on the semisimplicity of group algebras of characteristic not dividing the order of the group 13.21. The Brauer group and the Skolem-Noether theorem are indicated in exercises, and proved more generally for separable algebras over commutative rings in Chapter 32.

Chapter 11

Tensor Products and Flat Modules

Let R be a ring, M a right and N a left R-module. As usual, $M \times N$ denotes cartesian product. If G is an abelian group, and $g : M \times N \to G$ is a mapping of sets, then for each $y \in N$ there is a mapping $g_y : M \to G$, defined by the formula $g_y(a) = g(a, y)\ \forall a \in M$. Symmetrically, if $x \in M$, then $g_x : N \to G$ is defined by the formula $g_x(b) = g(x, b)$ $\forall b \in M$. A mapping $g : M \times N \to G$ is **bilinear** in case $g_y : M \to G$ and $g_x : N \to G$ are group homomorphisms $\forall x \in M$, $y \in N$. A **balanced** map $g : M \times N \to G$ is a bilinear map such that

$$g(xr, y) = g(x, ry)$$

$\forall x \in M$, $y \in N$, $r \in R$.

Consider the category $T(M, R, N)$, consisting of balanced mappings $M \times N \to G$, where G is an abelian group. An initial object in this category is uniquely determined up to a unique isomorphism, and is denoted $M \underset{R}{\otimes} N$, or $M \otimes_R N$, read M **tensor** N (over R), also called the **tensor product** of M and N (over R). When the context is clear, we