II. Structure of Noetherian Semiprime Rings

The first thing to be said about Part II is that it does not depend upon the whole of Part I. We append here a list of prerequisites (not to be taken too literally!), and recommend the rereading of the Introduction, esp. pp. XI—IV, for an overview. Also reread the first paragraph of Suggestions for Reading This Book (p. XXI).

Minimal Background

0. Foreword (Cartesian products, and injective, surjective mappings)
1. Chapter 1 (Categories)
2. Chapter 2 (Direct sum and product)
3. Chapter 3
   (a) Free module R^n (3.9)
   (b) Matrix rings R_n (3.16 and 3.17)
   (c) The canonical isomorphisms (3.38, 3.39, and 3.40)

$$\mathrm{Hom}_R(X, A \oplus B) \approx \mathrm{Hom}_R(X, A) \oplus \mathrm{Hom}_R(X, B)$$
$$\mathrm{Hom}_R(A \oplus B, X) \approx \mathrm{Hom}_R(A, X) \oplus \mathrm{Hom}_R(B, X)$$

for any three R-modules X, A, and B

(d) The canonical isomorphism $\mathrm{End}_R A^n \approx E_n$, where $E = \mathrm{End}_R A$, and where $A^n = A \oplus \cdots \oplus A$ (n-summands) (3.17)

(f) Schur's lemma [3.10.1 (a) and (b); cf. 3.14.3]

(g) Definition of projective module and characterization as a summand of a free module (3.11.2)

(h) The notion of essential extension (3.58)

(i) (Optional) Injective Hulls (3.59)

4. (Optional) Chapter 4
   (a) Correspondence theorem (4.7)
   (b) Morita theorem (4.29)

Introduction to Part II

If M is an R-module with endomorphism ring K, there is a ring homomorphism

$$\delta : R \to \mathrm{End}_K M$$

($\delta = \delta_M$), where

$$x\delta(a) = xa$$

$\forall x \in M,\ a \in R$. A **general Wedderburn theorem** is a statement that δ_M is an epimorphism. In this case, the module M is said to be **balanced.**

A ring R is **simple** in case R contains no ideals other than 0 and R. The Wedderburn-Artin theorem implies that if R is a simple ring with minimum condition on right ideals, then any minimal right ideal I of R is balanced (as an R-module). Together with Schur's lemma stating that the endomorphism ring R' of I is a field, the theorem implies that R is the endomorphism ring of a vector space. Since the minimum condition on R implies that M has finite dimension n over R', that is, that $R'^n \approx M$ (as left R'-modules), then there is a ring isomorphism $R \approx R'_n$.

An R-module M is a generator in case for each R-module N there is an index set J and an epic $M^{(J)} \to N$. (It is equivalent to say that $M^n \to R \to 0$ is exact for some n; that is, $M^n \approx R \oplus A$, for some n and R-module A.)

In 1958, Morita proved a general Wedderburn theorem: *Every generator M in* mod-R *is balanced.* Furthermore, a balanced module M is a generator if and only if M is finitely generated and projective over End M_R (7.3). The proof of this is elementary and requires very little background from Part I.

How does this apply to simple rings? *If R is simple, then every non-zero right ideal I of R is a generator hence balanced.* For RI is an ideal, therefore $RI = \sum_{a\in R} aI = R$, hence there exist $a_i \in R$, $i = 1, \dots, n$, such that $\sum_{i=1}^{n} a_i I = R$ and, hence, a surjective mapping $I^n \to R$. This shows that general Wedderburn theorems are quite independent of the traditional chain conditions. The Wedderburn-Artin theorem is an immediate special case.

A ring satisfying the d.c.c. (a.c.c.) on right ideals is said to be right Artinian (Noetherian). Thus, simple right Artinian rings are isomorphic to matrix rings D_n over fields. The ring D_n is semisimple in the sense that every right module over it is a direct sum of simple modules. The full Wedderburn-Artin theorem (8.8) determines all semisimple rings as finite products of full matrix rings over fields, and these coincide with the right Artinian having no nilpotent ideals $\neq 0$.

The Wedderburn-Artin theorem is preceded in Chapter 7 by the corresponding result when R is a simple right Noetherian ring. In this case, a lemma states that I can be chosen so that $K = \text{End}\, I_R$ is an domain with right quotient field D, and R itself has a classical right quotient ring isomorphic to a matrix ring D_n over D for a unique integer $n > 0$. This latter result is a special case of a general theorem proved in Chapter 9, which states that a ring R has a classical right quotient ring Q that is a semisimple ring if and only if R has no nilpotent ideals $\neq 0$ and R satisfies the a.c.c. on annihilator and complement right ideals. Chapter 9 is devoted to the proof of this theorem. (Another proof is sketched in Chapter 16.)

A theorem in Chapter 10 determines all subrings R of a semilocal ring $Q = D_n$ such that Q is a classical right quotient ring of R. (Then R is said to be right order in Q.) The right orders of $Q = D_n$ are just the subrings containing K_n, where K is a right order in D (10.15 and 10.19). In case Q is a finite product of matrix rings over semilocal rings, then R contains a finite product of matrix rings over right orders in the semilocal rings.

Rings which are right orders in semilocal quotient rings D_n are characterized in Chapter 18 (e.g. 18.48), and maximal orders in the rings D_n are shown to be endomorphism rings of torsion free modules over right orders of D (10.29).

In Chapter 7, the Hilbert basis theorem states that the polynomial ring $R[x]$ (power series ring $R\langle x\rangle$) is right Noetherian if R is. The ring $A[y, D]$ of differential polynomials with respect to a derivation D of a ring A is also right Noetherian when A is (7.28). Moreover, if A is simple of characteristic 0, and if D is not an inner derivation, then $A[y, D]$ is also a simple ring. In case A is a field which is universal for the derivation D, then $A[y, D]$ is a right Noetherian V-ring in the sense that every simple module is injective (7.42). This ring has a simple injective cogenerator, but the ring itself is not semisimple. Furthermore, every submodule has a maximal submodule, and every cyclic module $\neq R$ is semisimple and injective. Every right V-ring having a semisimple classical right quotient ring is a finite product of simple right V-rings (7.36).

Over semisimple rings, every module is projective, and hence the global dimension of these rings is 0. The global dimension of the polynomial ring $R[x_1, \ldots, x_n]$ over a ring R is equal to n plus the global dimension of R. If R is semisimple, then $R[x_1, \ldots, x_n]$ has global dimension n (Hilbert Szyzgy Theorem 8.16). The Auslander theorems (8.19 to 8.21) assert

$$\text{r.gl.dim}\, R = \sup_{I \subseteq R}\{\text{proj. dim}\, R/I\},$$

and when R is not semisimple

$$\text{r.gl.dim}\, R = 1 + \sup_{I \subseteq R}\{\text{proj. dim}\, I\}.$$

Thus, R has right global dimension 1 if and only if every right ideal is projective. Simple Noetherian rings of global dimension not exceeding 2 are similar to Ore domains (8.27).