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Logical Number Theory I

An Introduction

With 13 Figures

Springer-Verlag
Berlin Heidelberg NewYork
London Paris Tokyo
Hong Kong Barcelona
Budapest

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Cover picture: It does not seem supercrogatory to inform the reader that the cover illustration is derived from the frontispiece to the first edition of Christopher Marlowe's biography of Dr. Johannes Faustus, whose insightful remarks on logic are quoted in this book.

Mathematics Subject Classification (1980): 03-01, 03F30, 10B99, 10N05

ISBN-13: 978-3-540-52236-2 e-ISBN-13: 978-3-642-75462-3
DOI: 10.1007/978-3-642-75462-3

Library of Congress Cataloging-in-Publication Data. Smoryński, C. Logical number theory: an introduction / Craig Smoryński. p. cm. - (Universitext) Includes bibliographical references and indexes. ISBN 3-540-52236-0 (Springer-Verlag Berlin: v. 1). - ISBN 0-387-52236-0 (Springer-Verlag New York: v. 1) 1. Number theory. 2. Logic, Symbolic and mathematical. I. Title. QA241.S614 1991 512'.7-dc20 90-25702

41/3140-543210 - Printed on acid-free paper

Preface

> What would life be like without arithmetic, but a scene of horrors?
>
> — *Rev. Sydney Smith*

It is with a mixture of trepidation and chutzpah that I offer a work bearing the audacious title of *Logical Number Theory*. Disadvantageous comparisons with the spectacularly successful *analytic* number theory are inevitable. It is a fact, however, that logicians have made their own studies of arithmetic, that, although not as deep or as fully developed as analytic number theory, there is a body of logical knowledge of the subject. The present work concerns itself with this fledgling logical study of number theory.

The present work, of which this book is the first of two volumes, is an *introductory* text. This means several things. First, it means that certain advanced and non-central topics have not been covered. Adequate expositions of some of these subjects already exist, and others are in preparation. Chapter VII of the second volume will offer references and recommendations to the reader who wishes to go further. For the present, let it suffice for me to say that these two volumes are intended to present the *core* of logical number theory and they do not go *much* beyond the basics. (I have, of course, indulged myself occasionally and included some items not clearly (or: clearly not) of central importance. I have done this whenever I felt the intruded topic to be of sufficient interest and brevity.)

A second characteristic of an introductory text should be a reasonably wide accessibility. It has been my intention that the present work be accessible to advanced undergraduate mathematics majors with minimal specific background knowledge as prerequisite. I presuppose that "mathematical maturity" which can no longer be presumed to be acquired from an American calculus course (the situation is better in Europe), a course in abstract algebra (always a good prerequisite for any serious logic course), and perhaps a little elementary number theory (if such was not covered in one's abstract algebra course) or a tiny bit of logic (say that available from the symbolic logic course offered by the philosophers). I emphasise that I do not wish to presuppose any amount of mathematical logic. For one thing, the arithmetical content of these two volumes ought to be of some interest to the general mathematician— who, in all probability, has had no prior contact with mathematical logic. To presuppose any logical background would thus lessen the book's potential usefulness. I do assume, however, that the reader has seen the connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$) and quantifiers ($\forall$, $\exists$), has an intuitive grasp of their meanings ("not", "and", "or", "implies" and "for all", "there exists", respectively), and will accept the linguistic condition that, if one wants to say, for example, that all objects x have property P, one writes this as

$$\forall x P(x),$$

and not as

$$P(x), \forall x,$$

this latter common usage having serious ambiguity if $P(x)$ is logically complex. (Think of the distinction between continuity and uniform continuity.)

I might mention at this point that the inclusion of logical material for the benefit of curious non-logicians has not *greatly* increased the size of the book. Because of the necessity, in the second volume, of verifying the derivability within formal systems of arithmetic of basic results of mathematical logic, all the basics have had to be discussed anyway. The addition of explanatory remarks for a wider audience did not take up much extra space. Indeed, the logical material that could be deleted on assumption of a prior logic course amounts to less than a single chapter.

The present work has proved somewhat longer than anticipated and is, thus, being split into two volumes. It is an entirely fortuitous circumstance that this, the first volume of the pair, is a complete work in itself. This volume could, in fact, be used as a slightly unorthodox textbook for an undergraduate introduction to mathematical logic. It covers almost everything I consider necessary for the "logical literacy" of the professional mathematician— the elements of recursion theory, the completeness theorem relating syntax and semantics, and the basic theorems on incompleteness and undecidability in their now traditional recursion theoretic formulations. What is missing is nonstandard analysis as well as some basic set theory, neither of which is a standard topic in introductory courses on mathematical logic anyway. The Table of Contents should give the potential instructor a hint of the extent of my unorthodoxy— the extra emphasis on arithmetic matters and the deëmphasis of pure logic: there is, for example, no section on the propositional calculus and truth tables. Such logical material (e.g., disjunctive normal form) is introduced as needed and omitted (e.g., duality) as it is not needed. Admitting such omissions, I must say, however, that I would not hesitate to use my book as an introductory logic textbook in a mathematics department. My book addresses the logical requisites of the general mathematician, not the future logician. (The latter can pick up whatever material I have overlooked in courses offered by the philosophy or informatics faculties, logicians not being as xenophobic as mathematicians in general.)

It was, of course, not my goal to write an introductory logic text, but, insofar as the present volume could serve as one, I offer a few suggestions on its use for such a purpose. First, given all the time in the world, I would cover the whole book. With a little less time I would omit the starred sections (in the case of I.9 omission is not completely advisable— skim it). More realistically, one could cover only Chapters I and III and have a solid core of logic. Pressed for time, a bare-bones course in logic could be given by I.3, I.7 - 12 (skimming I.9), III.2, III.3, and III.6 (plus III.7 or III.8 as time permits). All the central results would be covered.

As I said, my goal is not an introduction to logic, but to logical number theory. It would be rather pointless to use this volume in a course and not cover anything from Chapter II. The quickest route to the unsolvability of Hilbert's tenth problem is to be had by reading II.1, II.2, II.4, and II.5. The applications of II.6 and II.7 are, however, very nice and these sections ought to be at least minimally perused. Professional number theorists should read the whole of Chapter II; those with no prior background in recursion theory will need also I.7, I.8, I.10, I.11, and a bit of I.12. Number theorists who aren't familiar with the

Fueter-Pólya Theorem might also find I.4 - 5 to their liking. It presupposes III.2, but the material of III.4 may also be of some interest to the specialist.

Such dissections as above are compromises, distasteful to any author who, quite naturally, views his work as a whole. In the present case the whole consists of two volumes and I should like to say a few words about their overall plan. The simplest description is to be had by appeal to the unifying theme of undecidability: Chapter I introduces the theory of computability and offers abstract unsolvable problems; Chapter II shows that unsolvability is is not totally abstract by offering a concrete unsolvable problem. With the introduction of formal languages, Chapter III changes the nature of undecidability. Instead of offering so-called *mass problems* with no uniform effective solutions, it exhibits— *abstractly*— the existence of individual problems which cannot be decided on the basis of one's current set of axioms. It yields results like: "there are sentences which can neither be proven nor refuted", or, via Chapter II, "there are Diophantine equations possessing no solutions but for which there is no proof that they possess no solutions". Chapter IV begins the concretisation of such formal undecidability by presenting Gödel's original, deeper results. Not only can formal theories not decide everything, but there are specific assertions of intelligible content that cannot be decided. The first of these is the assertion of the consistency of a given formal theory. This is Gödel's Second Incompleteness Theorem and it is proven in Chapter IV of volume II. In Chapter VI, we will see the unprovability of specific combinatorial theorems in a strong formal theory of arithmetic. This proof uses nonstandard models of arithmetic, which are introduced and studied in Chapter V.

The above description is a gross oversimplification, but it will do for now as it illustrates the dichotomy between the two volumes: volume I covers the basics (at an advanced undergraduate level) with which all mathematicians ought to be partially familiar; and volume II covers more sophisticated material (at the first-year graduate level) with which all logicians ought to be partially familiar.

More years ago than should be clear from the final product, I was approached by the editors of the series *Perspectives in Mathematical Logic* to write a book on the "metamathematics of arithmetic". During the early stages of this work I received much financial support from the Heidelberg Academy of Sciences through the offices of Professor Doctor Gert H. Müller. Although the present book is not the same advanced monograph originally contracted for the *Perspectives*, it is the ultimate fruit of the seed planted by the editors of that series and I take the opportunity here to acknowledge both my conceptual and financial indebtedness.

While some of the writing in the earlier stages of the preparation of this work was performed in Heidelberg, most of the writing took place at the Rijksuniversiteit te Utrecht in the Netherlands under the hospitality of my dear friend Dirk van Dalen. This is particularly true of volume II, Chapters V and VI of which are based on the lecture notes of a course I gave in Utrecht during the Fall of 1978.

For their more direct involvement, I wish to thank my friend Friedemann Tuttas and the editorial staff of Springer-Heidelberg. The former proofread the typescript and made many

corrections and suggestions for improvement; the latter have always been a pleasure to deal with.

Finally, I would like to express my gratitude to my friends Julia Robinson (now deceased) and Petr Hájek for their encouragement over the years.

Westmont, Illinois (USA)
June 1990 *Craig Smoryński*

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