Grundlehren der mathematischen Wissenschaften 294

A Series of Comprehensive Studies in Mathematics

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Quadratic and Hermitian Forms over Rings

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona

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Mathematics Subject Classification (1980):
10C05, 13-XX, 14F05, 14F15, 16-XX, 18F25, 18G50, 20GXX

ISBN-13: 978-3-642-75403-6 e-ISBN-13: 978-3-642-75401-2
DOI: 10.1007/ 978-3-642-75401-2

Library of Congress Cataloging in Publication Data.
Knus, Max-Albert. Quadratic and hermitian forms over rings/Max-Albert Knus.
p. cm. - (Grundlehren der mathematischen Wissenschaften; 294)
Includes bibliographical references and index.
ISBN-13: 978-3-642-75403-6
1. Forms, Quadratic. 2. Hermitian forms. 3. Commutative rings. I. Title. II. Series.
QA243.K55 1991 512'.74 - dc20 90-42429 CIP

Softcover reprint of the hardcover 1st edition 1991

Typesetting: Macmillan India, Ltd., Bangalore, India
41/3140-543210 Printed on acid-free paper

Foreword

From its birth (in Babylon?) till 1936 the theory of quadratic forms dealt almost exclusively with forms over the real field, the complex field or the ring of integers. Only as late as 1937 were the foundations of a theory over an arbitrary field laid. This was in a famous paper by Ernst Witt. Still too early, apparently, because it took another 25 years for the ideas of Witt to be pursued, notably by Albrecht Pfister, and expanded into a full branch of algebra.

Around 1960 the development of algebraic topology and algebraic K-theory led to the study of quadratic forms over commutative rings and hermitian forms over rings with involutions. Not surprisingly, in this more general setting, algebraic K-theory plays the role that linear algebra plays in the case of fields.

This book exposes the theory of quadratic and hermitian forms over rings in a very general setting. It avoids, as far as possible, any restriction on the characteristic and takes full advantage of the functorial aspects of the theory. The advantage of doing so is not only aesthetical: on the one hand, some classical proofs gain in simplicity and transparency, the most notable examples being the results on low-dimensional spinor groups; on the other hand new results are obtained, which went unnoticed even for fields, as in the case of involutions on 16-dimensional central simple algebras.

The first chapter gives an introduction to the basic definitions and properties of hermitian forms which are used throughout the book.

Chapter II distills the categorical aspects of hermitian forms. The beginning is rather formal but towards the end the reader is rewarded by very explicit and concrete results about hermitian forms on projective varieties. This, by the way, is one of the two single places where the author departs from affine schemes.

Chapter III recalls the theory of descent and introduces the cohomological tools that are needed in the subsequent chapters. A good portion of it is dedicated to Azumaya algebras, thus paving the way for the study of Clifford algebras in Chapter IV. In the latter, following, I believe, an idea of Martin Kneser, semiregular forms are introduced, to replace regular forms of odd rank when 2 is not invertible.

Clifford algebras are used in Chapter V to study forms of rank up to 6. An interesting tool here is the so called reduced pfaffian, which is to forms of rank 5 or 6, what the reduced norm of a quaternion algebra is to forms of rank 3 or 4.

The next two chapters, VI and VII, have a distinctive K-theoretic flavour. Chapter VI proves stability- and cancellation theorems in the linear case and the analogous results in the unitary case. It contains, in particular, a complete, detailed and (most probably) correct proof, due to Maria Saliani, of injective stability for the unitary group.

The results of Chapter VI are applied to polynomial rings in Chapter VII, which studies the quadratic analogues of the theorem of Quillen and Suslin on projective modules. The last chapter contains results on the Witt group of regular rings and some amusing computations of Witt groups of curves and surfaces.

Authors are frequently accused by nasty reviewers of inadmissible omissions, so let me indulge for a while in this pleasure. Where did the author leave L-theory? Doesn't he think that arithmetic also deserves a little place here and there? Is he, by any chance, totally unaware of all the recent results about quadratic bundles over projective varieties? And what about lattices, packings and codes? Well, this is like bemoaning the absence of canard à l'orange in a vegetarian restaurant. Human life and endurance being limited, the author has chosen to stress the algebraic aspects of the theory and to avoid – within reason – the overlapping with other remarkable books on quadratic forms, like those of Lam, Milnor-Husemoller and Scharlau. Within these limits his book gives a clear, detailed and complete picture of the theory of quadratic and hermitian forms.

Finally, two words about prerequisites. Although the level of the exposition is variable and tends to increase towards the last chapters, unproved theorems are always quoted, unless they are basic results in commutative algebra or homological algebra. Occasionally, some familiarity with algebraic geometry may be helpful, as in the proof of Horrocks' theorem, without being indispensable. As I said before, algebraic K-theory is both a tool and a source of inspiration, hence a previous knowledge of its more elementary results will allow the reader to proceed faster through Chapter VI and VII.

Summer 1990 I. Bertuccioni

Acknowledgements

This volume would not have been written without the help of my friend Manuel Ojanguren. Even if I am responsible for the final draft, his influence is ubiquitous and I hardly can give him sufficient credit for his ideas and advice. Many results presented in this book were developed over years by (or in collaboration with) R. Parimala, R. Sridharan and M. Ojanguren. I am deeply grateful to them for many wonderful hours doing mathematics in Bombay, Lausanne or Zurich.

Special thanks go to my former teacher Beno Eckmann for his continued encouragement, to I. Bertuccioni, who offered to write the foreword, and to Springer-Verlag for publishing the volume.

I enjoyed the help of many other friends and colleagues, to mention only some: E. Bayer-Fluckiger, M. Kneser, M. Kolster, A. Paques, H. G. Quebbemann, C. Riehm, W. Scharlau and C. B. Thomas. Further, I received assistance from former students: C. Arlettaz-Brack, W. Bichsel, B. Keller, T. Mittelholzer, Y. Pitteloup, M. Saliani, N. Spriano, M. Schneider. Mrs. D. Spörri typed the first version, Mrs. P. Keller read the final version and corrected some of my worst "french" english. It is a pleasure to thank all of them.

Conventions

A ring will always be associative with an identity element 1. Modules will be right modules and maps are written on the left like functions. The composition of two maps f, g is denoted by fg (first g, then f). The commutative base ring of an algebra will be usually denoted by R and unadorned tensor products are taken over R. The ring of $(n \times n)$-matrices over an algebra A is denoted by $M_n(A)$ and the group of units of A by $A^\bullet$.

Fall 1990 M.-A. Knus

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