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Continuity, Integration and Fourier Theory

Springer-Verlag Berlin Heidelberg New York
London Paris Tokyo

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Mathematics Subject Classification (1980): 26-XX, 28Axx, 42Axx

ISBN-13: 978-3-540-50017-9 e-ISBN-13: 978-3-642-73885-2
DOI: 10.1007/978-3-642-73885-2

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2141/3140-543210 - Printed on acid-free paper

Preface

This book is a textbook for graduate or advanced undergraduate students in mathematics and (or) mathematical physics. It is not primarily aimed, therefore, at specialists (or those who wish to become specialists) in integration theory, Fourier theory and harmonic analysis, although even for these there might be some points of interest in the book (such as for example the simple remarks in Section 15). At many universities the students do not yet get acquainted with Lebesgue integration in their first and second year (or sometimes only with the first principles of integration on the real line). The Lebesgue integral, however, is indispensable for obtaining a familiarity with Fourier series and Fourier transforms on a higher level; more so than by using only the Riemann integral. Therefore, we have included a discussion of integration theory – brief but with complete proofs – for Lebesgue measure in Euclidean space as well as for abstract measures. We give some emphasis to subjects of which an understanding is necessary for the Fourier theory in the later chapters. In view of the emphasis in modern mathematics curricula on abstract subjects (algebraic geometry, algebraic topology, algebraic number theory) on the one hand and computer science on the other, it may be useful to have a textbook available (not too elementary and not too specialized) on the subjects – classical but still important to-day – which are mentioned in the title of this book.

The book consists of four parts. The first part (three chapters) is devoted to a discussion of simple properties of (real or complex) continuous functions on k -dimensional space, followed by the theorems of Korovkin and Stone-Weierstrass. The third chapter contains the elementary theory of Fourier series of continuous functions. The sections on the Stone-Weierstrass theorem can be omitted if desired. In the second part (two chapters) we deal with integration and with L_p -spaces. Convolutions and approximate identities in these spaces receive appropriate attention in view of their importance for what follows. The third part consists of two chapters, one on Fourier series of Lebesgue summable functions and one on the Fourier transform.

The emphasis is on convergence (pointwise or in the sense of Cesaro or Abel summability). The space L_2 receives the special attention it deserves (theorems of Riesz-Fischer and Plancherel). In the fourth part we present several applications. Some of these are of a somewhat more advanced nature, such as those in the section on functions of analytic type and the one on the Hausdorff-Young theorem. The treatment of the heat and wave equations is in the spirit of the discussion in the Dym-McKean book.

The book contains more than sixty exercises, mainly in the second and third parts, some with hints for the solution. At several places, in particular in the sections on Fourier series and Fourier transforms, these exercises form an essential addition to the theory.

Throughout the book familiarity with basic elementary facts in mathematical analysis and linear algebra is assumed.

For further study we refer to the extensive treatises by A. Zygmund (mainly on Fourier series) and R.E. Edwards (Fourier series only), the briefer books by Y. Katznelson and H. Helson (both on an advanced level) and, for those interested in various applications, the fine book by H. Dym and H.P. McKean. The paper by W.A.J.B. Coppel contains an interesting survey of the history of Fourier theory. Finally, the recent book by A. Torchinsky on real-variable methods in harmonic analysis leads the reader to the frontiers of modern research in this area. Details about these books and about the papers mentioned in the text can be found in the list of references at the end of the book.

I take pleasure in thanking my wife Ada for having taken upon her the work of transforming my hand-written pages into a typed manuscript on the basis of which the final camera-ready manuscript was prepared.

Leiden, November 1988

Adriaan C. Zaanen

Contents

Chapter 1. The Space of Continuous Functions	1
1. Subsets of \mathbb{R}^k	1
2. The Space of Continuous Functions	6
3. Lattice Properties of the Space of Real Continuous Functions	14
Chapter 2. Theorems of Korovkin and Stone-Weierstrass	21
4. Theorems of Korovkin and Weierstrass	21
5. The Stone-Weierstrass Theorem	27
6. The Complex Stone-Weierstrass Theorem	36
Chapter 3. Fourier Series of Continuous Functions	39
7. Trigonometric Polynomials	39
8. Fourier Series	46
9. Fejér's Theorem on Uniform Convergence of Cesaro Means	52
10. Fourier Coefficients and Orders of Magnitude	56
Chapter 4. Integration and Differentiation	65
11. Measure	65
12. Integral	76
13. Product Integral, Fubini's Theorem and Convolution	90
14. Differentiation of the Integral	98
15. Measurability, Continuity and Differentiability	107
Chapter 5. Spaces L_p and Convolutions	111
16. Hölder's Inequality	111
17. Spaces L_p	115
18. Convolution	128
19. Convolution and Approximate Identities	132

Chapter 6. Fourier Series of Summable Functions	137
20. Fourier Coefficients and the Fourier Transform	137
21. Pointwise Convergence of Cesaro Means and Abel Means	142
22. Pointwise Convergence of Fourier Series	148
23. Hilbert Space and the Space L_2	160
Chapter 7. Fourier Integral	171
24. Some Useful Integrals	171
25. Inverse Fourier Transform	176
26. Convergence of the Inverse Fourier Transform	178
27. The Plancherel Theorem	188
Chapter 8. Additional Results	197
28. The Wilbraham-Gibbs Phenomenon	197
29. Absolute Convergence	199
30. Positive Definite Functions	202
31. Equidistribution of Sequences	206
32. Functions of Analytic Type	208
33. The Hausdorff-Young Theorem	214
34. The Poisson Sum Formula	224
35. The Heat Equation	228
36. More on the Heat Equation	234
37. The Wave Equation	242
References	247
Subject Index	249