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Differential Inclusions

Set-Valued Maps and Viability Theory

With 29 Figures

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This book is dedicated to
Anne-Laure, Claudia and Francesca

Epigraph

Why a book on differential inclusions?

There is a great variety of motivations that led mathematicians to study dynamical systems having velocities not uniquely determined by the state of the system, but depending loosely upon it, i.e., to replace differential equations

$$x' = f(x)$$

by differential inclusions

$$x' \in F(x)$$

when F is the set-valued map that associates to the state x of the system the set of feasible velocities.

Each one of these motivations offered a partial and biased view of differential inclusions, but all together contributed to the creation of a wealth of problems to a subject whose vitality is at present beyond doubts.

The first purpose of this book is to report on the common tools and ideas which were devised and proposed by those who were attracted by this field either for its own intrinsic interest and beauty or for its potential for applications in different fields.

But, besides this array of mathematical and physical motivations, social and biological sciences should provide many instances of differential inclusions. Indeed, if deterministic models are quite convenient for describing systems that arise in physics, mechanics, engineering and even, in microeconomics, their use for explaining the evolution of what we shall call "macrosystems" does not take in account the *uncertainty* (which, in particular, involves the impossibility of a comprehensive description of the dynamics of the system), the absence of *controls* (or the ignorance of the laws relating the controls and the states of the system) and the *variety* of available dynamics. These are reasons why usual dynamical systems, or even controled dynamical systems, may not be suitable for describing the evolution of states of systems derived from economics, social and biological sciences. This is our hope to supply scientists of these fields with an adequate tool. To justify this hope, we shall provide in this book an application to economics.

What are the problems which arise?

Naturally, as always in mathematics, we shall begin by studying the existence of solutions to several classes of differential inclusions and study the properties of the set of trajectories.

We may expect this set of trajectories to be rather large: hence a second class of problems consists naturally in devising mechanisms for selecting special trajectories.

A first class of such mechanisms is provided by *Optimal Control Theory:* it consists in selecting trajectories that optimize a given criterion, a functional on the space of all such trajectories.

This implicitely requires that:

1) there exists one decision maker who "controls" the system

2) such a decision maker has a perfect knowledge of the future (which is involved in the definition of the criterion)

3) the optimal trajectories are chosen once and for all at the origin of the period of time.

These requirements are not satisfied by the "macrosystems" that evolve according to the laws of Darwinian evolution.

Such macrosystems appear to have neither aims nor targets nor desire to optimize some criterion. But they face a minimal requirement, called *viability*, which is to remain "alive" in the sense of satisfying given binding constraints.

For that, they use a policy, *opportunism*, that enables the system to conserve viable trajectories that its lack of determinism - the *availability of several feasible velocities* - allows to find.

This provides a mathematical metaphor of this deep intuition of Democritus, "Everything that exists in the universe is due to chance and necessity".

This second class of mechanisms is the object of *Viability Theory*, which is thouroughly investigated in the second half of this book. In particular, we shall apply Viability Theory in the framework of Control Theory for *regulating* systems through *feedback controls.*

Paris, Venezia, J.-P. Aubin
in Spring 1984 A. Cellina

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The etching with the "putto" inscribing $\frac{dP}{dy}$ is taken from a 1787 edition of Euler's *Institutiones Calculi Differentiali.*

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