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# Differential Inclusions

Set-Valued Maps and Viability Theory

With 29 Figures



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*This book is dedicated to  
Anne-Laure, Claudia and Francesca*



pigraph

*Why a book on differential inclusions?*

There is a great variety of motivations that led mathematicians to study dynamical systems having velocities not uniquely determined by the state of the system, but depending loosely upon it, i.e., to replace differential equations

$$x' = f(x)$$

by differential inclusions

$$x' \in F(x)$$

when  $F$  is the set-valued map that associates to the state  $x$  of the system the set of feasible velocities.

Each one of these motivations offered a partial and biased view of differential inclusions, but all together contributed to the creation of a wealth of problems to a subject whose vitality is at present beyond doubts.

The first purpose of this book is to report on the common tools and ideas which were devised and proposed by those who were attracted by this field either for its own intrinsic interest and beauty or for its potential for applications in different fields.

But, besides this array of mathematical and physical motivations, social and biological sciences should provide many instances of differential inclusions. Indeed, if deterministic models are quite convenient for describing systems that arise in physics, mechanics, engineering and even, in microeconomics, their use for explaining the evolution of what we shall call "macrosystems" does not take in account the *uncertainty* (which, in particular, involves the impossibility of a comprehensive description of the dynamics of the system), the absence of *controls* (or the ignorance of the laws relating the controls and the states of the system) and the *variety* of available dynamics. These are reasons why usual dynamical systems, or even controlled dynamical systems, may not be suitable for describing the evolution of states of systems derived from economics, social and biological

sciences. This is our hope to supply scientists of these fields with an adequate tool. To justify this hope, we shall provide in this book an application to economics.

*What are the problems which arise?*

Naturally, as always in mathematics, we shall begin by studying the existence of solutions to several classes of differential inclusions and study the properties of the set of trajectories.

We may expect this set of trajectories to be rather large: hence a second class of problems consists naturally in devising mechanisms for selecting special trajectories.

A first class of such mechanisms is provided by *Optimal Control Theory*: it consists in selecting trajectories that optimize a given criterion, a functional on the space of all such trajectories.

This implicitly requires that:

- 1) there exists one decision maker who “controls” the system
- 2) such a decision maker has a perfect knowledge of the future (which is involved in the definition of the criterion)
- 3) the optimal trajectories are chosen once and for all at the origin of the period of time.

These requirements are not satisfied by the “macrosystems” that evolve according to the laws of Darwinian evolution.

Such macrosystems appear to have neither aims nor targets nor desire to optimize some criterion. But they face a minimal requirement, called *viability*, which is to remain “alive” in the sense of satisfying given binding constraints.

For that, they use a policy, *opportunism*, that enables the system to conserve viable trajectories that its lack of determinism – the *availability of several feasible velocities* – allows to find.

This provides a mathematical metaphor of this deep intuition of Democritus, “Everything that exists in the universe is due to chance and necessity”.

This second class of mechanisms is the object of *Viability Theory*, which is thoroughly investigated in the second half of this book. In particular, we shall apply Viability Theory in the framework of Control Theory for *regulating* systems through *feedback controls*.

Paris, Venezia,  
in Spring 1984

J.-P. Aubin  
A. Cellina

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The etching with the “putto” inscribing  $\frac{dP}{dy}$  is taken from a 1787 edition of Euler’s *Institutiones Calculi Differentiali*.

# Table of Contents

<b>Introduction</b> . . . . .	1
<b>Chapter 0. Background Notes</b> . . . . .	9
Introduction . . . . .	9
1. Continuous Partitions of Unity. . . . .	9
2. Absolutely Continuous Functions . . . . .	12
3. Some Compactness Theorems . . . . .	13
4. Weak Convergence and Asymptotic Center of Bounded Sequences . . . . .	15
5. Closed Convex Hulls and the Mean-Value Theorem . . . . .	18
6. Lower Semicontinuous Convex Functions and Projections of Best Approximation. . . . .	21
7. A Concise Introduction to Convex Analysis . . . . .	29
<b>Chapter 1. Set-Valued Maps</b> . . . . .	37
Introduction . . . . .	37
1. Set-Valued Maps and Continuity Concepts . . . . .	39
2. Examples of Set-Valued Maps . . . . .	46
3. Continuity Properties of Maps with Closed Convex Graph . . . . .	54
4. Upper Hemicontinuous Maps and the Convergence Theorem . . . . .	59
5. Hausdorff Topology . . . . .	65
6. The Selection Problem . . . . .	68
7. The Minimal Selection . . . . .	70
8. Chebishev Selection . . . . .	73
9. The Barycentric Selection . . . . .	77
10. Selection Theorems for Locally Selectionable Maps . . . . .	80
11. Michael's Selection Theorem . . . . .	82
12. The Approximate Selection Theorem and Kakutani's Fixed Point Theorem . . . . .	84
13. $\sigma$ -Selectionable Maps. . . . .	86
14. Measurable Selections . . . . .	90
<b>Chapter 2. Existence of Solutions to Differential Inclusions</b> . . . . .	93
Introduction . . . . .	93
1. Convex Valued Differential Inclusions. . . . .	96



2. Qualitative Properties of the Set of Trajectories of Convex-Valued Differential Inclusions . . . . .	103
3. Nonconvex-Valued Differential Inclusions . . . . .	111
4. Differential Inclusions with Lipschitzean Maps and the Relaxation Theorem . . . . .	119
5. The Fixed-Point Approach . . . . .	127
6. The Lower Semicontinuous Case . . . . .	134
<b>Chapter 3. Differential Inclusions with Maximal Monotone Maps . . . . .</b>	<b>139</b>
Introduction . . . . .	139
1. Maximal Monotone Maps . . . . .	140
2. Existence and Uniqueness of Solutions to Differential Inclusions with Maximal Monotone Maps . . . . .	147
3. Asymptotic Behavior of Trajectories and the Ergodic Theorem . . . . .	151
4. Gradient Inclusions . . . . .	158
5. Application: Gradient Methods for Constrained Minimization Problems . . . . .	163
<b>Chapter 4. Viability Theory: The Nonconvex Case . . . . .</b>	<b>172</b>
Introduction . . . . .	172
1. Bouligand's Contingent Cone . . . . .	176
2. Viable and Monotone Trajectories . . . . .	179
3. Contingent Derivative of a Set-Valued Map . . . . .	188
4. The Time Dependent Case . . . . .	191
5. A Continuous Version of Newton's Method . . . . .	195
6. A Viability Theorem for Continuous Maps with Nonconvex Images . . . . .	198
7. Differential Inclusions with Memory . . . . .	204
<b>Chapter 5. Viability Theory and Regulation of Controlled Systems: The Convex Case . . . . .</b>	<b>213</b>
Introduction . . . . .	213
1. Tangent Cones and Normal Cones to Convex Sets . . . . .	218
2. Viability Implies the Existence of an Equilibrium . . . . .	228
3. Viability Implies the Existence of Periodic Trajectories . . . . .	235
4. Regulation of Controlled Systems Through Viability . . . . .	238
5. Walras Equilibria and Dynamical Price Decentralization . . . . .	245
6. Differential Variational Inequalities . . . . .	264
7. Rate Equations and Inclusions . . . . .	274
<b>Chapter 6. Liapunov Functions . . . . .</b>	<b>281</b>
Introduction . . . . .	281
1. Upper Contingent Derivative of a Real-Valued Function . . . . .	284
2. Liapunov Functions and Existence of Equilibria . . . . .	290

3. Monotone Trajectories of a Differential Inclusion . . . . . 293  
4. Construction of Liapunov Functions . . . . . 305  
5. Stability and Asymptotic Behavior of Trajectories . . . . . 309  
**Comments** . . . . . 322  
**Bibliography** . . . . . 328  
**Index** . . . . . 341