Die Grundlehren der mathematischen Wissenschaften

in Einzeldarstellungen
mit besonderer Berücksichtigung
der Anwendungsgebiete

Band 181

J. L. Lions · E. Magenes

Non-Homogeneous Boundary Value Problems and Applications

Translated from the French by
P. Kenneth

Volume I

Springer-Verlag Berlin Heidelberg New York 1972

J. L. Lions
University of Paris

E. Magenes
University of Pavia

Title of the French Original Edition:
Problèmes aux limites non homogènes et applications (tome I)
Publisher: S. A. Dunod, Paris 1968

Translator:

P. Kenneth

Paris

AMS Subject Classifications (1970)

Primary 35J20, 35J25, 35J30, 35J35, 35J40, 35K20, 35K35, 35L20

Secondary 46E35

ISBN-13:978-3-642-65163-2 e-ISBN-13:978-3-642-65161-8

DOI: 10.1007/978-3-642-65161-8

Library of Congress Catalog Card Number 71-151407

Preface

1. We describe, at first in a very formal manner, our essential aim. Let $\mathcal{O}$ be an open subset of $\mathbf{R}^n$, with boundary $\partial\mathcal{O}$. In $\mathcal{O}$ and on $\partial\mathcal{O}$ we introduce, respectively, linear differential operators

$$P \text{ and } Q_j, \quad 0 \leqq j \leqq \nu.$$

By "non-homogeneous boundary value problem" we mean a problem of the following type: let f and g_j, $0 \leqq j \leqq \nu$, be given in function spaces F and G_j, F being a space "on $\mathcal{O}$" and the G_j's spaces "on $\partial\mathcal{O}$"; we seek u in a function space $\mathcal{U}$ "on $\mathcal{O}$" satisfying

$$Pu = f \quad \text{in} \quad \mathcal{O}, \tag{1}$$

$$Q_j u = g_j \quad \text{on} \quad \partial\mathcal{O}, \quad 0 \leqq j \leqq \nu^{((1))}. \tag{2}$$

Q_j may be identically zero on part of $\partial\mathcal{O}$, so that the number of boundary conditions may depend on the part of $\partial\mathcal{O}$ considered[2].

We take as "working hypothesis" that, for $f \in F$ and $g_j \in G_j$, the problem (1), (2) admits a unique solution $u \in \mathcal{U}$, which depends continuously on the data[3].

But for *all linear* problems, there is a large number of *choices* for the spaces $\mathcal{U}$ and $\{F; G_j\}$ (naturally linked together).

Generally speaking, our aim is to determine families of spaces $\mathcal{U}$ and $\{F; G_j\}$, associated in a "natural" way with problem (1), (2) and convenient for applications, and also *all* possible choices for $\mathcal{U}$ and $\{F; G_j\}$ in these families.

Let us make this explicit by means of two examples, chosen as the simplest possible ones, but which already demonstrate the utility of non-homogeneous problems.

((1)) The Q_j's will be called "boundary operators". Such problems are called *non-homogeneous* because if we consider the setting of *unbounded operators*, then $Pu = f$, $u \in D(P)$ (= domain of P) implies null boundary conditions; hence a certain difference between f and the boundary data g_j.

[2] This will obviously be the case for most problems of evolution.

[3] At least in general; for elliptic problems uniqueness conditions will not be satisfied, but in any case we shall deal with operators *with indices* and therefore still have uniqueness on passing to the quotient by finite-dimensional subspaces. We shall verify that this "working hypothesis" is satisfied in each particular situation.

2. Examples

2.1 For $\mathcal{O}$, we take an open subset Ω of $\mathbf{R}^n$, with boundary Γ, and for P the operator Δ, Δ = Laplacian; we take $\nu = 0$ and Q_0 = identity. Then, the problem corresponding to (1), (2) is the classical *Dirichlet problem for* Δ:

$$\Delta u = f \text{ in } \Omega, \tag{3}$$

$$u = g_0 = g \text{ on } \Gamma. \tag{4}$$

We then ask *in what spaces f and g may be chosen so that* (3), (4) *admits a unique solution* (*in an appropriate sense*).

Classical answers are furnished by potential theory and the "Dirichlet principle": for example, we may choose f, g and certain of their derivatives in Ω and Γ respectively to be square integrable and obtain u, the solution of (3), (4), as well as certain of its derivatives to be square integrable in Ω.

Therefore, a *natural* family for problem (3), (4) must be (if we limit ourselves to "L^2 theory", that is Hilbert theory) the *Hilbert family of the Sobolev spaces* $H^s(\Omega)$ and $H^s(\Gamma)$, where $H^s(\Omega)$ (resp. $H^s(\Gamma)$) is, if s is an integer $\geqq 0$, the space of u's such that u and its derivatives (in the sense of distributions) up to order s are square integrable in Ω (resp. Γ) (this definition is generalized to all *real* s by introducing the derivative of order s by Fourier transform; see Chapter 1).

Here is one of the results we shall prove (see Chapter 2):

Let s be any real, non-negative number; if $f \in H^s(\Omega)$ and $g \in H^{s+3/2}(\Gamma)$, then there exists a unique $u \in H^{s+2}(\Omega)$ which is a solution of (3), (4) (*having given an appropriate sense to* (3) *and* (4) *separately, by a natural generalization of the classical definitions*). *More precisely*: *the operator $u \to \{\Delta u, u|_\Gamma\}$ is an isomorphism of*

$$H^{s+2}(\Omega) \quad \text{onto} \quad H^s(\Omega) \times H^{s+3/2}(\Gamma).$$

It must be pointed out that the derivatives of non-integer order *necessarily* enter the problem if we want the optimal result for each s, since $s + 2$ and $s + \frac{3}{2}$ cannot be integers simultaneously!

Furthermore, it is equally natural to study the case "negative s", since many problems deal with (3), (4), where, for example, with $f = 0$, g is *very irregular*: such as g square integrable on Γ (this is the case for *optimal control theory*), or g = the Dirac-mass at the point $x_0 \in \Gamma$ (then the solution u yields the Poisson kernel of the problem) or more generally g = an arbitrary distribution on Γ.

It is still possible to solve problem (3), (4) when s is a *negative real* number, the spaces $\mathcal{U}$ and G remaining of the same type (i.e. $\mathcal{U} = H^{s+2}(\Omega)$, $G = H^{s+3/2}(\Gamma)$, but with negative s) and the space F being an appropriate subspace $\Xi^s(\Omega)$ of $H^s(\Omega)$ consisting of elements which do not grow too rapidly "in the neighborhood of Γ" (see Chapter 2, Sections 6 and 7).

It follows that (3), (4) is solvable with g an *arbitrary distribution* on Γ, since then g necessarily belongs to a space $H^s(\Gamma)$, for an appropriate s.

In fact, in volume 3 of this book, we shall see that g may belong to the space of analytic functionals on Γ (and this space is the most general for which, at least for $f = 0$, it is possible to *give meaning to problem* (3), (4)).

2.2 As a second example, we consider the *heat operator*

$$P = \frac{\partial}{\partial t} - \Delta_x$$

in

$$\mathcal{O} = \Omega \times]0, T[\subset \mathbf{R}^{n+1};$$

the part of the boundary $\partial\mathcal{O}$ on which boundary conditions are given splits up into

$$\bar{\Omega} \quad \text{and} \quad \Sigma = \Gamma \times]0, T[.$$

Then, a problem corresponding to (1), (2) is

(5) $$\frac{\partial u}{\partial t} - \Delta_x u = f \quad \text{in} \quad \mathcal{O},$$

(6) $$u(x, 0) = u_0(x) \quad \text{in} \quad \Omega,$$

(7) $$u = g \quad \text{on} \quad \Sigma.$$

One of our aims is to obtain the largest possible families of spaces for f, u_0 and g such that (5), (6), (7) admits a unique solution, in an appropriate sense. We shall see, in Chapter 4 of Volume 2, that a "natural" Hilbert family of spaces $\mathcal{U}$ for the solution is the family $H^{2s,s}(\mathcal{O})$, with s any real number, where $H^{2s,s}(\mathcal{O})$ is, if s is a non-negative integer, the space of u's such that u and its derivatives up to order s in t and up to order $2s$ in x are square integrable in $\mathcal{O}$. The family $H^{2s,s}(\mathcal{O})$ plays, for problem (5), (6), (7), an analogous role to the family $H^s(\Omega)$ for problem (3), (4). Also, remarks analogous to those we made for problem (3), (4) are valid for problem (5), (6), (7).

3. We now specify which are the principal systems $\{P, Q_j\}$ studied in this book.

3.1 We consider the case where P is an *elliptic* operator (denoted by A) and the Q_j's are normal boundary operators (denoted B_j), where A and B_j verify suitable ellipticity conditions (see Chapter 2).

3.2 We consider the case where

$$P = \frac{\partial}{\partial t} + A,$$

a parabolic operator with suitable boundary conditions (Chapters 3 and 4).

3.3 We also consider the cases

$$P = \frac{\partial^2}{\partial t^2} + A$$

and

$$P = \frac{\partial}{\partial t} + \mathrm{i}\,A,$$

where A is a self-adjoint elliptic operator, and still with suitable boundary conditions (Chapters 3 and 5).

4. For all these problems, we *proceed systematically as follows* (except for possibly different techniques):

(i) we study the *regularity* of problem (1), (2), i.e.: assuming the data f and g_j to be *regular* (in a sense to be specified), we study the *corresponding regularity of* u;

(ii) by *transposition* of (i) (for the "adjoint problem") we deduce therefrom (with a suitable technique and in particular the obtainment of "*trace theorems*") the solution of problem (1), (2) for data belonging to spaces of distributions;

(iii) by *interpolation* between (i) and (ii), we obtain "*intermediate*" results.

Of course, the systematic setting-up of such a program is an enormous task and many possibilities had to be put aside (we have formulated them in lists of problems in the last sections of each chapter).

In general, we consider for (i):

in volumes 1 and 2: data which are finitely often *differentiable* in the sense of L^2 (spaces such as $H^s(\Omega)$, $H^{2s,s}(\mathcal{O})$, $H^s(\Gamma)$, ...)

in volume 3: *analytic data or data belonging to suitable Gevrey classes.*

5. As we have seen, the present volume depends on regularity theorems in "*differentiable in the sense of* L^2" spaces (*Sobolev spaces*).

Therefore, the *basic tools* are:

— Sobolev spaces constructed on L^2,
— the theory of interpolation of corresponding spaces.

This is the subject of Chapter 1, where we study interpolation only *for the Hilbert cases*; the introduction of interpolation between (non-

"hilbertizable") Banach spaces and its applications to Sobolev spaces constructed on L^p, $p \neq 2$, would have complicated this work considerably.

Once in possession of these tools we need to prove regularity theorems (stage (i)) and then to implement stages (ii) and (iii).

This is done for the situations described in Section 3, above. Let us be more precise.

The elliptic case is the subject of Chapter 2. Stage (i) is studied completely by the method of J. Peetre [*2*], under the hypotheses that A is *properly elliptic* and that the B_j's *cover* A in the sense of Lopatinskii-Shapiro and Agmon-Douglis-Nirenberg.

Stages (ii) and (iii) follow our previous papers on these subjects: see Lions-Magenes [*1*], [*2*] and [*3*] (where we also study the L^p case, for $1 < p < \infty$, which we disregard here for $p \neq 2$).

"Variational" evolution operators and their applications are studied in Chapter 3.

Partial differential equations of evolution are studied in more detail in Chapters 4 and 5 of Volume 2 and applications to optimal control theory in Chapter 6 of Volume 2. Other applications will be given in Volume 3.

6. We have made an effort to make the book readable in "local" fashion; indications about the logical relations between the different subjects are given at the beginning of each chapter.

7. Each chapter ends with a section of *comments* and a section of *problems*.

The comments give bibliographical indications, which, although numerous, by no means cover the subject. This is especially the case for research work cited in the comments but not studied in this book.

The rather large number of problems to which we call attention are very unequal in difficulty. For cases where *results of type* (i) *are already available*, the execution of stages (ii) and (iii) may offer great technical difficulties if one looks for optimal results, but is certainly much more accessible if one is satisfied with results in the "neighborhood" of the optimal results. Situations for which the results of type (i) are lacking (and we indicate a number of such problems) may, of course, be much more difficult.

We wish to thank C. Baiocchi, M. S. Baouendi and G. Geymonat for reading various parts of the manuscript and giving us their comments.

Paris/Pavia, July 1967

J. L. Lions E. Magenes

Preface to the English Translation

The present translation follows the French edition without change, except for some corrections which were suggested to us by the remarks of M. S. Baouendi, G. Geymonat, C. Goulaouic and P. Schapira, to all of whom we express our sincerest thanks. We have added a complementary bibliography. We also wish to thank P. Kenneth for his excellent work of translation.

Paris/Pavia, October 1971

J. L. Lions   E. Magenes

Contents

Chapter 1

Hilbert Theory of Trace and Interpolation Spaces

Chapter 2

Elliptic Operators. Hilbert Theory

Chapter 3

Variational Evolution Equations

Contents of Volume II

Contents of Volume III