Timothy O'Meara was born on January 29, 1928. He was educated at the University of Cape Town and completed his doctoral work under Emil Artin at Princeton University in 1953. He has served on the faculties of the University of Otago, Princeton University and the University of Notre Dame. From 1978 to 1996 he was provost of the University of Notre Dame. In 1991 he was elected Fellow of the American Academy of Arts and Sciences.

O'Meara's first research interests concerned the arithmetic theory of quadratic forms. Some of his earlier work – on the integral classification of quadratic forms over local fields – was incorporated into one of the chapters of this, his first book.

Later research focused on the general problem of determining the isomorphisms between classical groups. In 1968 he developed a new foundation for the isomorphism theory which in the course of the next decade was used by him and others to capture all the isomorphisms among large new families of classical groups. In particular, this program advanced the isomorphism question from the classical groups over fields to the classical groups and their congruence subgroups over integral domains.

In 1975 and 1980 O'Meara returned to the arithmetic theory of quadratic forms, specifically to questions on the existence of decomposable and indecomposable quadratic forms over arithmetic domains.

O. Timothy O'Meara

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In Memory of my Parents

Preface

The main purpose of this book is to give an account of the fractional and integral classification problem in the theory of quadratic forms over the local and global fields of algebraic number theory. The first book to investigate this subject in this generality and in the modern setting of geometric algebra is the highly original work *Quadratische Formen und orthogonale Gruppen* (Berlin, 1952) by M. EICHLER. The subject has made rapid strides since the appearance of this work ten years ago and during this time new concepts have been introduced, new techniques have been developed, new theorems have been proved, and new and simpler proofs have been found. There is therefore a need for a systematic account of the theory that incorporates the developments of the last decade.

The classification of quadratic forms depends very strongly on the nature of the underlying domain of coefficients. The domains that are really of interest are the domains of number theory: algebraic number fields, algebraic function fields in one variable over finite constant fields, all completions thereof, and rings of integers contained therein. Part One introduces these domains via valuation theory. The number theoretic and function theoretic cases are handled in a unified way using the Product Formula, and the theory is developed up to the Dirichlet Unit Theorem and the finiteness of class number. It is hoped that this will be of service, not only to the reader who is interested in quadratic forms, but also to the reader who wishes to go deeper into algebraic number theory and class field theory. In Part Two there is a discussion of topics from abstract algebra and geometric algebra which will be used later in the arithmetic theory. Part Three treats the theory of quadratic forms over local and global fields. The direct use of local class field theory has been circumvented by introducing the concept of the quadratic defect (which is needed later for the integral theory) right at the start. The quadratic defect gives, in effect, a systematic way of refining certain types of quadratic approximations. However, the global theory of quadratic forms does present a dilemma. Global class field theory is still so inaccessible that it is not possible merely to quote results from the literature. On the other hand a thorough development of global class field theory cannot be included in a book of this size and scope. We have therefore decided to compromise by specializing the methods of global class field theory to the case of quadratic extensions, thereby obtaining all that is needed for the global theory of quadratic forms. Part Four starts with a systematic development of the formal aspects of integral quadratic forms over Dedekind domains. These techniques are then applied, first to solve the local integral classification problem, then to investigate the global integral theory, in particular to establish the relation between the class, the genus, and the spinor genus of a quadratic form.

It must be emphasized that only a small part of the theory of quadratic forms is covered in this book. For the sake of simplicity we confine ourselves entirely to quadratic forms and the orthogonal group, and then to a particular part of this theory, namely to the classification problem over arithmetic fields and rings. Thus we do not even touch upon the theory of hermitian forms, reduction theory and the theory of minima, composition theory, analytic theory, etc. For a discussion of these matters the reader is referred to the books and articles listed in the bibliography.

O. T. O'MEARA

February, 1962.

I wish to acknowledge the help of many friends and mathematicians in the preparation of this book. Special thanks go to my former teacher EMIL ARTIN and to GEORGE WHAPLES for their influence over the years and for urging me to undertake this project; to RONALD JACOBOWITZ, BARTH POLLAK, CARL RIEHM and HAN SAH for countless discussions and for checking the manuscript; and to Professor F. K. SCHMIDT and the Springer-Verlag for their encouragement and cooperation and for publishing this book in the celebrated Yellow Series. I also wish to thank Princeton University, the University of Notre Dame and the Sloan Foundation[1] for their generous support.

O. T. O'MEARA

December, 1962.

[1] ALFRED P. SLOAN FELLOW, 1960—1963.

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Part Three

Arithmetic Theory of Quadratic Forms over Fields

Part Four

Arithmetic Theory of Quadratic Forms over Rings

Prerequisites and Notation

If X and Y are any two sets, then $X \subset Y$ will denote strict inclusion, $X - Y$ will denote the difference set, $X \rightarrow Y$ will denote a surjection of X onto Y, $X \rightarrowtail Y$ an injection, $X \rightarrowtail\!\!\!\rightarrow Y$ a bijection, and $X \twoheadrightarrow\!\!\!- Y$ an arbitrary mapping. By "almost all elements of X" we shall mean "all but a finite number of elements of X".

N denotes the set of natural numbers, Z the set of rational integers, Q the set of rational numbers, R the set of real numbers, P the set of positive numbers, and C the set of complex numbers.

We assume a knowledge of the elementary definitions and facts of general topology, such as the concepts of continuity, compactness, completeness and the product topology.

From algebra we assume a knowledge of 1) the elements of group theory and also the fundamental theorem of abelian groups, 2) galois theory up to the fundamental theorem and including the description of finite fields, 3) the rudiments of linear algebra, 4) basic definitions about modules.

If X is any additive group, in particular if X is either a field or a vector space, then $\dot{X}$ will denote the set of non-zero elements of X. If H is a subgroup of a group G, then $(G:H)$ is the index of H in G. If E/F is an extension of fields, then $[E:F]$ is the degree of the extension. The characteristic of F will be written $\chi(F)$. If α is an element of E that is algebraic over F, then $\operatorname{irr}(x, \alpha, F)$ is the irreducible monic polynomial in the variable x that is satisfied by α over the field F. If E_1 and E_2 are subfields of E, then E_1E_2 denotes the compositum of E_1 and E_2 in E. If E/F is finite, then $N_{E/F}$ will denote the norm mapping from E to F; and $S_{E/F}$ will be the trace.