Algorithms and Combinatorics 1

Study and Research Texts

Karl Heinz Borgwardt

The Simplex Method

A Probabilistic Analysis

With 42 Figures in 115 Separate Illustrations

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This book is dedicated

to

KARSTEN and STEFFEN

DORIS

MOTHER and FATHER

I had very little time for them while I was writing it.

PREFACE

For more than 35 years now, George B. Dantzig's Simplex-Method has been the most efficient mathematical tool for solving linear programming problems. It is probably that mathematical algorithm for which the most computation time on computers is spent. This fact explains the great interest of experts and of the public to understand the method and its efficiency. But there are linear programming problems which will not be solved by a given variant of the Simplex-Method in an acceptable time. The discrepancy between this (negative) theoretical result and the good practical behaviour of the method has caused a great fascination for many years. While the "worst-case analysis" of some variants of the method shows that this is not a "good" algorithm in the usual sense of complexity theory, it seems to be useful to apply other criteria for a judgement concerning the quality of the algorithm.

One of these criteria is the average computation time, which amounts to an analysis of the average number of elementary arithmetic computations and of the number of pivot steps. A rigid analysis of the average behaviour may be very helpful for the decision which algorithm and which variant shall be used in practical applications.

The subject and purpose of this book is to explain the great efficiency in practice by assuming certain distributions on the "real-world"-problems. Other stochastic models are realistic as well and so this analysis should be considered as one of many possibilities.

This book was written to collect and to summarize the ideas and results of several papers. I began with the analysis of the average complexity of the Simplex-Method in my dissertation under the advice of Professor H. Brakhage. I want to thank him for directing my mathematical interest towards this fruitful field of research and for many valuable discussions during that time.

My research on this subject has two aspects:

- the search for a theoretical approach

- the evaluation and estimation of rather difficult expectation values given as integrals.

The theoretical aspect consists of two parts

- finding a Phase II-algorithm which is appropriate for such an analysis (done in the dissertation 1977)
- finding a Phase I-algorithm which meets the necessary stochastic assumptions (done in 1981).

The evaluation turned out to be the greater problem. Only step by step could I obtain the desired results

- asymptotic bounds under special distributions (dissertation 1977), which have been improved significantly in this book
- upper and lower asymptotic bounds under rather general stochastic assumptions (1978, 1979 and 1984)
- polynomial upper bounds under general assumptions for Phase II and for the complete method (1981, improved 1984)
- polynomial upper bounds for the problem type with nonnegativity constraints (1984).

Some of the considerations and calculations are very lengthy, technical and complicated. For that reason I tried to explain in detail and to illustrate what I mean in a great number of figures.

The Introduction gives a survey over the most important developments in this field of research. It consists of four parts. In a first part the formulation of the problem and basic notation are introduced. After that we give a rather informal survey over the main developments in the analysis of the algorithm in the past. Part 3 deals with the question which stochastic model seems to be appropriate. Part 4 summarizes the following chapters, the methods, the results and the conclusions. Here the improvements in the results (compared with their original version) become apparent. This chapter may be interesting even for readers who are not interested in details. In Chapter I the Simplex-Method and the special variant used for our analysis are explained. Here I use an approach which differs from the usual terminology using "basic" and "nonbasic" variables. I hope that this part will be instructive even for people who are not familiar with that algorithm. Chapter II describes the stochastic model and requires elementary probability theory. Rather technical and lengthy are Chapters III and V. Here it is shown that the average number of steps is polynomial. For the proof some elementary techniques of integration in $\mathbb{R}^n$ are necessary. Chapter V shows that the results of Chapter III can be saved even when the assumption of rotational symmetry is weakened to a certain degree. In Chapter IV various methods for the analysis of the asymptotic behaviour are demonstrated. And the Appendix gives some formulae and estimations which are frequently used.

I want to thank Prof. B. Korte, Prof. L. Lovász and Prof. M. Grötschel for many valuable hints and Mrs. Th. Konnerth for the excellent typesetting.

Finally, I want to make two remarks. I have used the "we"-form in the book in order to include the reader into the considerations and to let him participate. And, of course, my English is not perfect. Please do not mind!

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