I. Bifurcation Theory

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Preface

The word "bifurcation" means "splitting into two". "Bifurcation" is used to describe any sudden change that occurs while parameters are being smoothly varied in any system: dynamical, ecological, etc. Our survey is devoted to the bifurcations of phase portraits of differential equations – not only to bifurcations of equilibria and limit cycles, but also to perestroikas of the phase portraits of systems in the large and, above all, of their invariant sets and attractors. The statement of the problem in this form goes back to A.A. Andronov.

Connections with the theory of bifurcations penetrate all natural phenomena. The differential equations describing real physical systems always contain parameters whose exact values are, as a rule, unknown. If an equation modeling a physical system is structurally unstable, that is, if the behavior of its solutions may change qualitatively through arbitrarily small changes in its right-hand side, then it is necessary to understand which bifurcations of its phase portrait may occur through changes of the parameters.

Often model systems seem to be so complex that they do not admit meaningful investigation, above all because of the abundance of the variables which occur. In the study of such systems, some of the variables that change slowly in the course of the process described are, as a rule, assumed to be constant. The resulting system with a smaller number of variables can then be investigated. However, it is frequently impossible to consider the individual influences of the discarded terms in the original model. In this case, the discarded terms may be looked upon as typical perturbations, and, accordingly, the original model can be described by means of bifurcation theory applied to the reduced system.

Reformulating the well-known words of Poincaré on periodic solutions, one may say that bifurcations, like torches, light the way from well-understood dynamical systems to unstudied ones. L.D. Landau, and later E. Hopf, using this idea of bifurcation theory, offered a heuristic description of the transition from laminar to turbulent flow as the Reynolds number increases. In Landau's scenario this transition was accomplished through bifurcations of tori of steadily growing dimensions. Later on when the zoo of dynamical systems and their bifurcations had significantly grown, many papers appeared, describing – mainly at a physical level – the transition from regular (laminar) flow to chaotic (turbulent) flow. The chaotic behavior of the 3-dimensional model of Lorenz for convective motions has been explained with the aid of a chain of bifurcations. This explanation is not included in the present survey since, to save space, bifurcations of systems with symmetry have not been included. Lorenz's system is centrally symmetric.

The theory of relaxation oscillations, which deals with systems in which the parameters slowly change with time (these parameters are called slow variables), closely adjoins the theory of bifurcations in which parameters do not change with time. In "fast-slow" systems of relaxation oscillations, a slowness parameter enters that characterizes the speed of change of the slow variables. When this parameter is zero, a fast-slow system transforms into a family studied in the theory of bifurcations, but at a nonzero value of the parameter specific phenomena arise which are sometimes called dynamical bifurcations.

In this survey, systematic use is made of the theory of singularities. The solutions to many problems of bifurcation theory (mostly of local ones) consist of presenting and investigating a so-called principal family – a kind of topological normal form for families of the class studied. The theory of singularities helps to guess at, and partially to investigate, principal families. This theory also describes the theory of bifurcations of equilibrium states, singularities of slow surfaces, slow motions in the theory of relaxation oscillations, etc.

We also note that finitely smooth normal forms of local families of differential equations are especially useful in the theory of nonlocal bifurcations. On one hand, these normal forms substantially simplify the presentation and investigation of bifurcations, and also simplify and clarify the proof and analysis of the results obtained. On the other hand, the nonlocal theory of bifurcations helps to select problems from the theory of normal forms that are important for applications. In our opinion, at the present time, the connection between the theory of normal forms and the nonlocal theory of bifurcations is not used often enough.

This survey includes, along with what is known, a series of new results, some of these are known to the authors through private communications. [Added in translation: The results mentioned below were new when the Russian text was written (1985). Now most of them have been published. The additional list of references is given after the main one and numbered.] Among these are eight new topics. The first is a complete investigation of bifurcations from equilibria in generic two-parameter families of vector fields on the plane with two intersecting invariant curves (the so-called reduced problem for two purely imaginary pairs, Sect. 4.5 and Sect. 4.6 of Chap. 1 (see Żołądek (1987)). The second is the construction of finitely smooth normal forms and functional moduli of the C^1-classification of local families of vector fields and diffeomorphisms (Yu.S. Il'yashenko and S.Yu. Yakovenko, Sect. 5.7–5.10 of Chap. 2 (see Il'yashenko and Yakovenko [3*, 4*])). The third is the construction of a topological invariant of vector fields with a trajectory homoclinic to a saddle with complex eigenvalues (Sect. 5.6 of Chap. 3). The fourth is the description of a generic two-parameter deformation of a vector field with two homoclinic curves at a saddle, in which the bifurcation diagram of the deformation contains a continuum of components. (D.V. Turaev and L.P. Shil'nikov [9*], Sect. 7.2 of Chap. 3). The fifth result is the definition of a statistical limit set as a possible candidate for the concept of a physical attractor (Sect. 8.2 of Chap. 3 (Il'yashenko [2*])). The sixth one is the description of connections between the theory of implicit equations and relaxation oscillations, and the normalization of slow motions for fast-slow systems with one or two slow variables (see Arnol'd's theorem in Sect. 2.2–2.7 of Chap. 4 and the related paper by Davidov [1*]). The seventh result is normalization of fast-slow equations, and the explicit form and investigation of systems of first approximation (Sect. 3.2–3.5 of Chap. 4; see the related paper by Teperin [8*]). The eighth and last one is the investigation of the delayed loss of stability in generic fast-slow systems as a pair of eigenvalues of a stable singular point of a fast equation crosses the imaginary axis (the birth of a cycle as a dynamical bifurcation (A.I. Nejshtadt, § 4 of Chap. 4); see [6*, 7*]). We also point here to a conjecture on the bifurcations in generic multiple parameter families of vector fields on the plane that is closely related to Hilbert's 16th problem (Sect. 2.8 of Chap. 3).

Our survey, inevitably, is incomplete. We did not include in it the comparatively few works on local bifurcations in three-parameter families and on nonlocal bifurcations in two-parameter families; some relevant citations are, however, given in the References. In describing nonlocal bifurcations we limited ourselves to only those things which happen on the boundary of the set of Morse-Smale systems. The theory of such bifurcations is substantially complete, although it is not very well known; it is mostly due to works of the Gor'kij school, which often have been published in sources that are hard to obtain. That part of the boundary of the set of Morse-Smale systems on which a countable set of nonwandering trajectories arise is not yet fully explored; but Sect. 7 of Chap. 3 is devoted to this problem. For reasons of consistency of style we often formulate known results in a form different from that in which they first appeared.

Chap. 1 and 2 were written by V.I. Arnol'd and Yu.S. Il'yashenko. Chap. 3, in its final version, was written by V.S. Afrajmovich and Yu.S. Il'yashenko with the participation of V.I. Arnol'd and L.P. Shil'nikov. Sect. 1.6 of Chap. 2 was written by V.S. Afrajmovich. Sects. 1 and 2 of Chap. 4 were written by V.I. Arnol'd, Sect. 3, except for Sect. 3.7, by Yu.S. Il'yashenko. Sect. 3.7 was written by N.Kh. Rozov, Sect. 4 by A.I. Nejshtadt, Sect. 5 by A.K. Zvonkin; the authors sincerely thank them. The authors do not claim that the list of References is complete. In its organization we followed the same principles as in the survey by Arnol'd and Il'yashenko (1985). The symbol ▲ denotes the end of some formulations.