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HANDBOOK OF ELLIPTIC INTEGRALS FOR ENGINEERS AND PHYSICISTS

by

PAUL F. BYRD

and

MORRIS D. FRIEDMAN

SPRINGER-VERLAG BERLIN HEIDELBERG GMBH

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WITH 22 FIGURES

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Preface.

Engineers and physicists are more and more encountering integrations involving nonelementary integrals and higher transcendental functions. Such integrations frequently involve (not always in immediately recognizable form) elliptic functions and elliptic integrals.

The numerous books written on elliptic integrals, while of great value to the student or mathematician, are not especially suitable for the scientist whose primary objective is the ready evaluation of the integrals that occur in his practical problems. As a result, he may entirely avoid problems which lead to elliptic integrals, or is likely to resort to graphical methods or other means of approximation in dealing with all but the simplest of these integrals.

It became apparent in the course of my work in theoretical aerodynamics that there was a need for a handbook embodying in convenient form a comprehensive table of elliptic integrals together with auxiliary formulas and numerical tables of values. Feeling that such a book would save the engineer and physicist much valuable time, I prepared the present volume.

Although the book is not a text, an attempt has been made to write it in elementary terms so that no previous knowledge of elliptic integrals, theta functions or elliptic functions is needed. A collection of over 3000 integrals and formulas, designed to meet most practical needs, is presented using Legendre's and Jacobi's notations, rather than the less familiar Weierstrassian forms. Many of these formulas are substitutions and recurrence relations for evaluating additional integrals which are not explicitly written. Sufficient explanatory material and cross-references are given to permit the reader to obtain the answers he requires with a minimum of effort.

Short tables of numerical values are given for the elliptic integrals of the first and second kind, for Jacobi's "nome" q, for the function denoted by Heuman as Λ_0, and for K times the Jacobian Zeta function. Tables of the last three functions are useful in the numerical evaluation of elliptic integrals of the third kind.

Particular precautions, of course, have to be taken in a work of this kind to insure accuracy of the formulas. My co-author, Mr. Morris

D. FRIEDMAN, undertook the job of verifying each formula. Where ever possible, they were either derived independently in different ways or checked against more than one source. Criticisms of the material contained in the handbook and notice of any errors which may yet appear in it will be sincerely welcomed.

It is impossible to acknowledge properly all the sources to which debt is owed. The bibliography, however, lists many books in which the derivation of some of the formulas can be found or where related material may be obtained. For friendly advice and valuable suggestions, I am under obligation to Professors A. ERDÉLYI, W. MAGNUS and R. C. ARCHIBALD, and to colleagues in the Theoretical Aerodynamics Section, Ames Aeronautical Laboratory, NACA. To my colleague, DORIS COHEN, I am grateful for a critical reading of the manuscript and for many suggestions leading to improvement of exposition and organization. Hearty thanks are also extended to Mrs. ROSE CHIN BYRD and Mrs. MARY T. HUGGINS for assistance in the task of preparing the tables of numerical values and to Mr. DUANE W. DUGAN for help in reading the proofs.

On behalf of both authors I wish finally to express gratitude to Springer-Verlag and to Professor K. KLOTTER of Stanford University who kindly called their attention to our work. The appearance of this book is in no small measure due to their cooperative attitude, their encouragement, and their genuine interest in the promotion of technical publications.

Palo Alto, California. PAUL F. BYRD.

July, 1953.

Table of Contents.

List of Symbols and Abbreviations.

The following table comprises a list of the principal symbols and abbreviations used in the handbook. Notations not listed are so well understood that explanation is unnecessary.

Symbol or Abbreviation	Meaning	Section
α^2	Parameter of elliptic integral of the third kind	**110**
$\operatorname{am}(u, k) \equiv \operatorname{am} u$	Amplitude u	**120**
$\operatorname{am}^{-1}(y, k)$	Inverse amplitude y	**130**
$\operatorname{cd} u$	$\equiv \dfrac{\operatorname{cn} u}{\operatorname{dn} u}$	**120**
$\operatorname{cd}^{-1}(y, k)$	—	**130**
$\operatorname{cn}(u, k) \equiv \operatorname{cn} u$	Cosine amplitude u; Jacobian elliptic function	**120**
$\operatorname{cn}^{-1}(y, k)$	—	**130**
$\cos^{-1}\varphi$	Inverse trigonometric function, often written *arc cos* φ	
$\operatorname{cs} u$	$\equiv \dfrac{\operatorname{cn} u}{\operatorname{sn} u}$	**120**
$\operatorname{cs}^{-1}(y, k)$	—	**130**
$\operatorname{dc} u$	$\equiv \dfrac{\operatorname{dn} u}{\operatorname{cn} u}$	**120**
$\operatorname{dc}^{-1}(y, k)$	—	**130**
$\operatorname{dn} u$	Delta amplitude u; Jacobian elliptic function	**120**
$\operatorname{dn}^{-1}(y, k)$	—	**130**
$\operatorname{ds} u$	$\equiv \dfrac{\operatorname{dn} u}{\operatorname{sn} u}$	**120**
$\operatorname{ds}^{-1}(y, k)$	—	**130**
e_1, e_2, e_3	Roots of polynomial written in Weierstrassian form	**1030**
$E(\varphi, k) \equiv E(u)$	Legendre's incomplete elliptic integral of the second kind; ($\varphi = \operatorname{am} u$)	
$E'(\varphi, k) \equiv E(\varphi, k')$	Associated incomplete elliptic integral of the second kind	
$E(k) \equiv E \equiv E(\pi/2, k)$	Complete elliptic integral of the second kind	**110** (bracketed: applies to $E(\varphi,k)$ through $F(\varphi,k)$)
$E' \equiv E(k')$	Associated complete elliptic integral of the second kind	
$F(\varphi, k) \equiv u$	Incomplete elliptic integral of the first kind	

Symbol or Abbreviation	Meaning	Section
$F(a, b; c; z)$	$\equiv \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} z^m$, hypergeometric series	900
G	$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \approx 0.91596559$, Catalan's constant	615
g_2, g_3	—	1030
$\Gamma(z)$	$\equiv \frac{1}{z} \prod_{m=1}^{\infty} \left[\left(1+\frac{1}{m}\right)^z \left(1+\frac{z}{m}\right)^{-1}\right]$, $z \neq 0, -1, -2, \ldots$, Gamma function	
H, H_1	Eta functions of Jacobi	1050
$I_\nu(z)$	$\equiv \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{\nu+2m}$ $\equiv e^{-\nu\pi i/2} J_\nu(iz)$, modified Bessel functions of first kind	560
Im	Imaginary part of a complex quantity	
$J_\nu(z)$	$\equiv \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{\nu+2m}$, Bessel functions of first kind	560
k	Modulus of Jacobian elliptic functions and integrals	110
$k' = \sqrt{1-k^2}$	Complementary modulus	110
$K(k) \equiv K = F(\pi/2, k)$	Complete elliptic integral of the first kind	110
$K' \equiv K(k')$	Associated complete elliptic integral of the first kind	110
$\Lambda_0(\varphi, k)$	$\equiv \frac{2}{\pi}[EF(\varphi, k') + KE(\varphi, k') - KF(\varphi, k')]$, Heuman's Lambda function	150
$\ln z$	Natural logarithm of z	
m, n	Integers, unless otherwise stated	
$n!$	$= 1.2 \ldots n$; n factorial	
$\mathrm{nc}\, u$	$\equiv \frac{1}{\mathrm{cn}\, u}$	120
$\mathrm{nc}^{-1}(y, k)$	—	130
$\mathrm{nd}\, u$	$\equiv \frac{1}{\mathrm{dn}\, u}$	120
$\mathrm{nd}^{-1}(y, k)$	—	130
$\mathrm{ns}\, u$	$\equiv \frac{1}{\mathrm{sn}\, u}$	120
$\mathrm{ns}^{-1}(y, k)$	—	130
$\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6$	—	430

Symbol or Abbreviation	Meaning	Section
$\wp(u)$	Weierstrassian elliptic function	**1030**
$\Pi(\varphi, \alpha^2, k) \equiv \Pi(u, \alpha^2)$	Legendre's incomplete elliptic integral of the third kind; ($\varphi = \operatorname{am} u$)	**110**
$\Pi(\alpha^2, k) \equiv \Pi(\pi/2, \alpha^2, k)$	Complete elliptic integral of the third kind	**110**
q	$\equiv e^{-\pi K'/K}$, referred to as the *nome*	**1050**
$Q_n(z)$	$\equiv \frac{1}{2^{n+1}} \int_{-1}^{1} (1-t^2)^n (z-t)^{-n-1}\, dt$, R.P. $(n+1) > 0$, Legendre functions (spherical harmonics)	**560**
R.P.	Real part of a complex quantity	
$\sigma(u)$	Weierstrassian Sigma function	**1030**
$\operatorname{sd} u$	$\equiv \frac{\operatorname{sn} u}{\operatorname{dn} u}$	**120**
$\operatorname{sd}^{-1}(y, k)$	—	**130**
$\sin^{-1}\varphi$	Inverse trigonometric function, often written *arc sin* φ	
$\operatorname{sn} u$	Sine amplitude u	**120**
$\operatorname{sn}^{-1}(y, k)$	—	**130**
$\operatorname{tn} u \equiv \operatorname{sc} u$	$\equiv \frac{\operatorname{sn} u}{\operatorname{cn} u}$	**120**
$\operatorname{tn}^{-1}(y, k)$	—	**130**
Θ, Θ_1	Jacobi's Theta functions	**1050**
$\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3$	Elliptic Theta functions	**1050**
y	Variable limit of integration in all integrals	
$\psi(z)$	$\equiv \xi - \frac{1}{z} + \sum_{m=1}^{\infty} \frac{z}{m(z+m)}$, digamma function; ξ is Euler's number ≈ 0.5772156665	**900**
$Z(u, k) \equiv Z(u) \equiv Z(\beta, k)$	$\equiv E(u) - \frac{E}{K} u$, Jacobian Zeta function; ($\operatorname{am} u = \beta$)	**140**
$\zeta(u)$	Weierstrassian Zeta function	**1030**
$(a)_n$	$= a(a+1)\dots(a+n-1)$, for $n = 1, 2, \dots$ $(a)_0 = 1$, Pochhammer's symbol	
$\binom{a}{n} = (-1)^n \frac{(-a)_n}{n!}$	$= \frac{a(a-1)\dots(a-n+1)}{1.2.3\dots n}$; $\binom{a}{0} = 1$.	