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Elliptic Functions

With 14 Figures



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*Dedicated to the memory of
Heinz and Anja Hopf*

When the familiar scene is suddenly strange
Or the well known is what we have yet to learn,
And two worlds meet, and intersect, and change;
T. S. Eliot

Preface

This book has grown out of a course of lectures on elliptic functions, given in German, at the Swiss Federal Institute of Technology, Zürich, during the summer semester of 1982. Its aim is to give some idea of the theory of elliptic functions, and of its close connexion with theta-functions and modular functions, and to show how it provides an analytic approach to the solution of some classical problems in the theory of numbers. It comprises eleven chapters. The first seven are function-theoretic, and the next four concern arithmetical applications. There are Notes at the end of every chapter, which contain references to the literature, comments on the text, and on the ramifications, old and new, of the problems dealt with, some of them extending into cognate fields. The treatment is self-contained, and makes no special demand on the reader's knowledge beyond the elements of complex analysis in one variable, and of group theory.

Professor Raghavan Narasimhan has read the definitive English version of the text, and made illuminating comments, as a result of which I have improved the presentation in several places. Dr. Anton Good has looked through the first German version, and spared the time for many useful discussions. Dr. Peter Thurnheer, who had attended the course, helped me check the detailed calculations that lurk behind some of the statements in the text. Mr. Albert Stadler has assisted me in tracing the bibliographical references and in proof-reading. My sincere thanks go to them all for sustaining this effort during a difficult twelvemonth.

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Zürich, October 1984

K. C.

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