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Representations of Preferences Orderings

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Preface

A basic assumption made by pioneers of classical microeconomics such as Edgeworth and Pareto was that the ranking of a consumer's preferences could always be measured numerically, by associating to each possible consumption bundle a real number that measured its *utility*: the greater the utility, the more preferred was the bundle, and conversely. It took several decades before the naivety of this assumption was seriously challenged by economists, such as Wold, attempting to find conditions under which it could be justified mathematically. Wold's work was the first in a long chain of results of that type, leading to the definitive theorems of Debreu and others in the 1960s, and subsequently to the refinements and generalisations that have appeared in the last twenty-five years.

Out of this historical background there has appeared a general mathematical problem which, as well as having applications in economics, psychology, and measurement theory, arises naturally in the study of sets bearing order relations:

> *Given some kind of ordering $\succsim$ on a set S, find a real-valued mapping u on S such that for any elements x, y of S, $x \succsim y$ if and only if $u(x) \geq u(y)$. If also S has a topology (respectively, differential structure), find conditions that ensure the continuity (respectively, differentiability) of the mapping u.*

A mapping u of this kind is called a *representation* of the ordering $\succsim$.

In this book we have tried to gather together, using a uniform, consistent terminology and notation, a number of the most important representation theorems which are scattered widely, with a corresponding variety of nomenclature, throughout the literature of several branches of mathematics. Our objective was to make those results accessible, in a more-or-less self-contained presentation, to readers with a basic knowledge of set theory, topology, measure theory, functional analysis, and differentiable manifolds, as found in [105], [139], [220], [226], and [22] respectively.[1] Our choice of topics was determined partly by our own preferences, partly by our lack of expertise in certain fields, and partly by our desire to make the book of

[1] A rudimentary knowledge of economics will provide motivation for some of the concepts in the book, but is not essential for the understanding of our results.

readable length. Whatever the omissions, we believe that we have included the most significant theorems and methods in the field.

We have assigned credit for results to the best of our knowledge. However, there are some ideas that are part of the folklore of the subject and have been included without attribution of credit. We acknowledge our indebtedness to the books of Fishburn [78], Roberts [216], and Davey and Priestley [64] in the preparation of Chapter 1.

The first drafts of this book were prepared using the T^3 *Scientific Word Processing System.* The final version was produced by converting the drafts to $\TeX$ and then using *Scientific Word.* T^3 and *Scientific Word* are both products of TCI Software Research, Inc.

NOTATION

We use the following notation for sets of numbers.

The set of natural numbers:	$\mathbf{N} \equiv \{0, 1, 2, \ldots\}$
The set of positive integers	$\mathbf{N}^+ \equiv \{1, 2, 3, \ldots\}$
The set of integers	$\mathbf{Z} \equiv \{0, \pm 1, \pm 2, \ldots\}$
The set of rational numbers:	$\mathbf{Q}$
The set of real numbers:	$\mathbf{R}$

We denote by $(x_n)_{n=0}^{\infty}$, or $(x_0, x_1, \ldots)$, or even just (x_n), the sequence whose terms are indexed by $\mathbf{N}$ and whose n^{th} term is x_n; sequences indexed by other sets, such as $\mathbf{N}^+$, are described similarly. For functions we use the arrow $\rightarrow$ as in 'the function $u : A \rightarrow B$', and the barred arrow $\mapsto$ as in 'the function $x \mapsto \sin x$ on $\mathbf{R}$'.

We call a set **countable** if its elements can be enumerated as a sequence $x_1, x_2, \ldots$ (possibly with repetitions), and **denumerable** if it is in one-one correspondence with $\mathbf{N}^+$.

We normally denote the metric in any metric space (and in particular, any normed linear space) by ρ, and the open (respectively, closed) ball with centre a and radius r by $B(a, r)$ (respectively, $\overline{B}(a, r)$). X° will denote the interior, and $\overline{X}$ the closure, of a set X in a topological space.

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Douglas Bridges and *Ghanshyam Mehta*
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