

# Lecture Notes in Economics and Mathematical Systems

422

## Founding Editors:

M. Beckmann  
H. P. Künzi

## Editorial Board:

H. Albach, M. Beckmann, G. Feichtinger, W. Hildenbrand, W. Krelle  
H. P. Künzi, K. Ritter, U. Schittko, P. Schönfeld, R. Selten

## Managing Editors:

Prof. Dr. G. Fandel  
Fachbereich Wirtschaftswissenschaften  
Fernuniversität Hagen  
Feithstr. 140/AVZ II, D-58097 Hagen, FRG

Prof. Dr. W. Trockel  
Institut für Mathematische Wirtschaftsforschung (IMW)  
Universität Bielefeld  
Universitätsstr. 25, D-33615 Bielefeld, FRG

Douglas S. Bridges   Ghanshyam B. Mehta

# Representations of Preferences Orderings



Springer

## Authors

Douglas S. Bridges  
University of Waikato  
Department of Mathematics and Statistics  
Hamilton, New Zealand

Ghanshyam B. Mehta  
University of Queensland  
Department of Economics  
Queensland 4072, Brisbane  
Australia

ISBN 978-3-540-58839-9

ISBN 978-3-642-51495-1 (eBook)

DOI 10.1007/978-3-642-51495-1

Library of Congress Cataloging-in-Publication Data. Bridges, D. S. (Douglas S.), 1945- . Representations of preference orderings / Douglas Bridges, Ghanshyam Mehta. p. cm. – (Lecture notes in economics and mathematical systems; 422) Includes bibliographical references and index.

I. Consumers' preferences—Mathematical models. I. Mehta, Ghanshyam. II. Title. III. Series. HF5415.3.B69 1995 658.8'343—dc20 9445869

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting, reproduction on microfilms or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag Berlin Heidelberg GmbH. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1995

Originally published by Springer-Verlag Berlin Heidelberg New York in 1995

Typesetting: Camera ready by author

SPIN: 10486729

42/3142-543210 - Printed on acid-free paper

# 1

## Preface

A basic assumption made by pioneers of classical microeconomics such as Edgeworth and Pareto was that the ranking of a consumer's preferences could always be measured numerically, by associating to each possible consumption bundle a real number that measured its *utility*: the greater the utility, the more preferred was the bundle, and conversely. It took several decades before the naivety of this assumption was seriously challenged by economists, such as Wold, attempting to find conditions under which it could be justified mathematically. Wold's work was the first in a long chain of results of that type, leading to the definitive theorems of Debreu and others in the 1960s, and subsequently to the refinements and generalisations that have appeared in the last twenty-five years.

Out of this historical background there has appeared a general mathematical problem which, as well as having applications in economics, psychology, and measurement theory, arises naturally in the study of sets bearing order relations:

*Given some kind of ordering  $\succeq$  on a set  $S$ , find a real-valued mapping  $u$  on  $S$  such that for any elements  $x, y$  of  $S$ ,  $x \succeq y$  if and only if  $u(x) \geq u(y)$ . If also  $S$  has a topology (respectively, differential structure), find conditions that ensure the continuity (respectively, differentiability) of the mapping  $u$ .*

A mapping  $u$  of this kind is called a *representation* of the ordering  $\succeq$ .

In this book we have tried to gather together, using a uniform, consistent terminology and notation, a number of the most important representation theorems which are scattered widely, with a corresponding variety of nomenclature, throughout the literature of several branches of mathematics. Our objective was to make those results accessible, in a more-or-less self-contained presentation, to readers with a basic knowledge of set theory, topology, measure theory, functional analysis, and differentiable manifolds, as found in [105], [139], [220], [226], and [22] respectively.<sup>1</sup> Our choice of topics was determined partly by our own preferences, partly by our lack of expertise in certain fields, and partly by our desire to make the book of

---

<sup>1</sup>A rudimentary knowledge of economics will provide motivation for some of the concepts in the book, but is not essential for the understanding of our results.

readable length. Whatever the omissions, we believe that we have included the most significant theorems and methods in the field.

We have assigned credit for results to the best of our knowledge. However, there are some ideas that are part of the folklore of the subject and have been included without attribution of credit. We acknowledge our indebtedness to the books of Fishburn [78], Roberts [216], and Davey and Priestley [64] in the preparation of Chapter 1.

The first drafts of this book were prepared using the  $T^3$  *Scientific Word Processing System*. The final version was produced by converting the drafts to  $T_E X$  and then using *Scientific Word*.  $T^3$  and *Scientific Word* are both products of TCI Software Research, Inc.

## NOTATION

We use the following notation for sets of numbers.

The set of natural numbers:	$\mathbf{N} \equiv \{0, 1, 2, \dots\}$
The set of positive integers	$\mathbf{N}^+ \equiv \{1, 2, 3, \dots\}$
The set of integers	$\mathbf{Z} \equiv \{0, \pm 1, \pm 2, \dots\}$
The set of rational numbers:	$\mathbf{Q}$
The set of real numbers:	$\mathbf{R}$

We denote by  $(x_n)_{n=0}^\infty$ , or  $(x_0, x_1, \dots)$ , or even just  $(x_n)$ , the sequence whose terms are indexed by  $\mathbf{N}$  and whose  $n^{\text{th}}$  term is  $x_n$ ; sequences indexed by other sets, such as  $\mathbf{N}^+$ , are described similarly. For functions we use the arrow  $\rightarrow$  as in ‘the function  $u : A \rightarrow B$ ’, and the barred arrow  $\mapsto$  as in ‘the function  $x \mapsto \sin x$  on  $\mathbf{R}$ ’.

We call a set **countable** if its elements can be enumerated as a sequence  $x_1, x_2, \dots$  (possibly with repetitions), and **denumerable** if it is in one-one correspondence with  $\mathbf{N}^+$ .

We normally denote the metric in any metric space (and in particular, any normed linear space) by  $\rho$ , and the open (respectively, closed) ball with centre  $a$  and radius  $r$  by  $B(a, r)$  (respectively,  $\overline{B}(a, r)$ ).  $X^\circ$  will denote the interior, and  $\overline{X}$  the closure, of a set  $X$  in a topological space.

## ACKNOWLEDGEMENTS

Douglas Bridges wishes to thank

the Kerr-Fry Foundation at the University of Edinburgh, and the Universities of Buckingham (England) and Waikato (New Zealand), for their support of visits by each author to the other's institution, thereby helping to minimise the number of problems arising in a collaboration over such great distances;

his coauthor for inviting him to participate in this project; and his wife and children for their continuing support and patience over a project that has taken many years to bring to fruition.

Ghanshyam Mehta wishes to thank

the late Professor L. Nachbin for his inspiration and encouragement—in particular, for suggesting that this book be written;

Professor G. Debreu for his inspiring lectures on mathematical economics at the University of California, Berkeley, and for his graciousness (Mehta's interest in order preserving functions originated in Professor Debreu's proof of the existence of a continuous utility function during a mathematical economics course);

A.F. Beardon, J.C. Candeal, G. Herden, E. Indurain, and P.K. Monteiro for many discussions about order preserving functions; Meena, Maithili, and Kunti for their love and encouragement; and

his mother Nandini and father Bhagvandas for their love and for inculcating a respect for learning and the pursuit of knowledge.

*Douglas Bridges and Ghanshyam Mehta*  
22 October 1994

# Contents

<b>Preface</b>	<b>v</b>
<b>1 Ordered Sets and Order Homomorphisms</b>	<b>1</b>
1.1 Introduction . . . . .	1
1.2 Order Homomorphisms on a Finite Set . . . . .	7
1.3 Order Homomorphisms on a Countable Set . . . . .	8
1.4 Order Homomorphisms on an Uncountable Set . . . . .	10
1.5 Two Fundamental Theorems of Cantor . . . . .	16
1.6 Order and Topology . . . . .	19
<b>2 Order Homomorphisms in Euclidean Space</b>	<b>27</b>
2.1 The Euclidean Distance Approach . . . . .	27
2.2 Neufeind's Measure-theoretic Construction . . . . .	34
<b>3 The Fundamental Theorems</b>	<b>37</b>
3.1 The Gap Lemma . . . . .	37
3.2 The Classical Theorems of Eilenberg and Debreu . . . . .	44
<b>4 A Miscellany of Representations</b>	<b>49</b>
4.1 Peleg's Theorem . . . . .	49
4.2 Jaffray's Open Gap Lemma . . . . .	52
4.3 Sondermann's Theorems . . . . .	54
4.4 The Eilenberg-Richter Method . . . . .	61
4.5 Order Homomorphisms on Ordered Topological Groups . . . . .	66
4.6 Order Isomorphisms on Topological Vector Spaces . . . . .	68
<b>5 The Urysohn-Nachbin Approach</b>	<b>73</b>
5.1 Introduction . . . . .	73
5.2 The Existence of Order Isomorphisms . . . . .	77
<b>6 Interval Orders</b>	<b>87</b>
6.1 Interval Orders and their Representation . . . . .	87
6.2 Strong Pseudotransitivity and Continuous Representations . . . . .	91
6.3 Representing Interval Orders in Euclidean Space . . . . .	94
6.4 Representing Interval Orders on a Topological Space . . . . .	102
6.5 Chateaufneuf's Representation Theorem . . . . .	104

<b>7</b>	<b>Differentiable Order Homomorphisms</b>	<b>109</b>
7.1	Preliminaries . . . . .	109
7.2	The Existence of Local Differentiable Order Isomorphisms .	113
7.3	The Existence of Differentiable Order Isomorphisms . . . . .	118
<b>8</b>	<b>Jointly Continuous Order Homomorphisms</b>	<b>125</b>
8.1	Introduction and Overview . . . . .	125
8.2	The Topology of Closed Convergence . . . . .	131
8.3	Levin's Theorem . . . . .	142
	<b>References</b>	<b>147</b>
	<b>Index</b>	<b>163</b>