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Serge Lang

Introduction to Modular Forms

With 9 Figures

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The AMS (MOS) classification scheme was made up before the subject of modular forms exploded. New numbers should be created for this subject.
It is impossible at present to find numbers fitting this book appropriately.

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Foreword

This book is intended as a partial survey for the elementary parts of an exceptionally active field which found a resurgence of interest over the last 8 years, after being almost forgotten for 30 years. I have attempted to put together some of the basic facts to make it easier for those who don't know the subject, to get some idea where it is going in the arithmetic direction, and how to get into it. I hope that the reader will find this book a helpful introduction to the Antwerp Conference volumes (Springer Lecture Notes).

It is unfortunate that Hecke's Institute Lecture Notes [H] never received wide distribution nor attention, and that they were omitted from his collected works. They summarize a great deal of his insights into modular forms. Ogg's book [O], for instance, follows almost the same table of contents, the main additions being the Petersson scalar product and Weil's theorem on functional equations, which Hecke did not have. Considering the progress which has been made since then, they have perhaps mostly historical interest, but I feel that even now, it is profitable to look at them. They have the merit, among many others, to be brief and accessible.

Partly because of Hitler and the war, which almost annihilated the German school of mathematics, and partly because of the great success of certain algebraic methods of Artin, Hasse, Deuring, modular forms and functions were to a large extent ignored by most mathematicians for about 30 years after the thirties. Eichler, Maass, Petersson, and Rankin were the main exceptions. It is striking that except for Petersson, the other three contributed to the International Colloquium on Zeta Functions, J. Indian Math. Soc. 1956. Maass was the first to develop a Hecke theory for non-holomorphic modular forms. In another direction Siegel in the 40's and 50's had some influence on the one variable case by his work on several variables, as well as through his Tata Institute notes. Selberg's contributions in the 50's were to have far reaching influence, but with some delays due to the lack of published proofs.

Taniyama, Shimura and Weil had much to do with bringing modular forms back into the forefront of mathematics. The Taniyama–Shimura conjecture relating modular forms of weight 2 and elliptic curves gave impetus to the subject. Langlands gave an exceedingly broad framework for the connection between modular forms and the arithmetic of number fields, involving what can be called non-abelian class field theory as a special case. He recognized the connection between Hecke's work on Dirichlet series associated with modular forms and the Artin L-functions of finite Galois extensions of the rationals, among others. In Jacquet-Langlands, it is shown how the Hecke theory can be viewed as a vast generalization of Kronecker's theorem that every abelian extension of the rationals is cyclotomic, modulo the "Artin conjecture" (that L-functions are entire), and the theory is seen to apply as well to not necessarily holomorphic modular forms. Conversely, it was proved by Serre and Deligne that to every holomorphic form of weight 1 it is possible to associate an "odd" 2-dimensional representation of the Galois group over the rationals.

Historically, it is very interesting that Hecke noticed explicitly that by the Mellin transform, one can associate a modular form to each entire function defined by a Dirichlet series having a functional equation of standard type with one gamma factor, and conversely. He was looking for such functions. At the same time and place that he was writing this, Artin was working with his L-series. But as Tate once said, neither was digging what the other was doing, and so they did not notice that they were doing two aspects of the same thing. One had to wait till the Langlands conjectures for that.

To me, it is this direction which motivates the study of modular forms, i.e. their connections with representations of Galois groups of number fields.

The contents of this book consists mostly of lectures given at Yale in fall 1974. The first two chapters are essential to everything that follows. On the other hand, the rest of the book can be read in sections which are independent of each other. The first half is organized around Hecke operators, in various settings, mostly for $SL_2(\mathbf{Z})$, and over the complex numbers, including work of Eichler–Shimura and Manin. The second half deals with p-adic properties and the connection with Galois groups due to Serre and Swinnerton–Dyer, and distribution theory according to Iwasawa, touching on the connection with values of zeta functions, and p-adic modular forms, as developed by, among others, Klingen, Siegel, Serre, Coates, Sinnott, Katz, Manin, Mazur, etc.

I tried to select topics for which no systematic introduction is yet available. Since several introductions are available for the connection between Dirichlet series with functional equations and modular forms, this topic has been omitted.

I am much indebted to Ribet, Serre, and Zagier for their careful reading of the manuscript.

New Haven, in Summer 1976. S. Lang

I have made no changes in this printing except for a number of corrections, the need for which was pointed out to me by many people, whom I thank.

The theory of modular forms has, of course, expanded enormously since the book was written, but I don't think these major developments have impaired the value of the book as an introduction. I leave to others the writing of books on the connection between modular forms, algebraic geometry, Iwasawa theory, and representation theory.

New Haven, 1995 Serge Lang

Table of Contents

Part III. Modular Forms for Congruence Subgroups

Part IV. Congruence Properties and Galois Representations

Part V. *p*-Adic Distributions