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The Computation of Fixed Points and Applications

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PREFACE

Fixed-point algorithms have diverse applications in economics, optimization, game theory and the numerical solution of boundary-value problems. Since Scarf's pioneering work [56,57] on obtaining approximate fixed points of continuous mappings, a great deal of research has been done in extending the applicability and improving the efficiency of fixed-point methods. Much of this work is available only in research papers, although Scarf's book [58] gives a remarkably clear exposition of the power of fixed-point methods. However, the algorithms described by Scarf have been superseded by the more sophisticated restart and homotopy techniques of Merrill [48,49] and Eaves and Saigal [14,16]. To understand the more efficient algorithms one must become familiar with the notions of triangulation and simplicial approximation, whereas Scarf stresses the concept of primitive set.

These notes are intended to introduce to a wider audience the most recent fixed-point methods and their applications. Our approach is therefore via triangulations. For this reason, Scarf is cited less in this manuscript than his contributions would otherwise warrant. We have also confined our treatment of applications to the computation of economic equilibria and the solution of optimization problems. Hansen and Koopmans [28] apply fixed-point methods to the computation of an invariant optimal capital stock in an economic growth model. Applications to game theory are discussed in Scarf [56,58], Shapley [59], and Garcia, Lemke and Luethi [24]. Allgower [1] and Jeppson [31] use fixed-point algorithms to find many solutions to boundary-value problems. Infinite-dimensional cases are also discussed by Freidenfelds [21] and Wilmuth [73]. The Schauder Projection theorem (see Freidenfelds [21] or Smart [61]) describes how a finite-dimensional approximation can be obtained. Our treatment is confined to finite-dimensional spaces throughout. More recent developments we have been unable to cover are the orientation theories of Shapley [60], Lemke and Grotzinger [44], Todd [66] and most generally Eaves [15] and Eaves and Scarf [17]; and the algorithmic improvements and convergence analysis of Saigal [53,54]. Eaves [15] contains a comprehensive bibliography.

The manuscript is organized as follows. Chapter I gives a classical (non-algorithmic) proof of Brouwer's theorem from Sperner's lemma, thus introducing the reader to several important concepts. A number of applications are described in Chapter II. We provide a formal treatment of triangulations in Chapter III with descriptions of some important particular triangulations. The latter are used in Chapter IV in algorithms for computing approximate fixed points of continuous functions. The applications of Chapter II motivate extensions from functions to point-to-set mappings. Chapters V and VI parallel Chapters I and II in proving and applying Kakutani's fixed-point theorem. Chapter VII describes an algorithm for computing Kakutani fixed points. This algorithm and those of Chapter IV are inefficient if a good approximation is desired. In Chapters VIII and IX we describe the more sophisticated restart and homotopy algorithms. The latter require special triangulations which are developed in Chapter X. Finally, Chapter XI describes some measures that can be used to compare different triangulations when used for fixed-point computation.

We have included a number of challenging exercises to increase the reader's understanding of the material. Some numerical examples of the algorithms are given in the text. The reader is assumed to be familiar with real analysis and linear programming, including lexicographic resolution of degeneracy. We also assume the Kuhn-Tucker conditions for nonlinear programming known.

This manuscript arose out of a course in computing fixed points that I gave at Cornell University in spring 1975. I am grateful to Michel Cosnard, Pierre Déjax, Pradeep Dubey, Etienne Loute, Shigeo Muto, Bob Rovinsky and Prakash Shenoy for preparing excellent notes. The National Science Foundation, through grant GK-42092, provided support during the preparation of this manuscript. I would like to thank Mrs. Kathy King for her excellent typing. Finally, my thanks go to my wife Marina for her encouragement and assistance.

Notation

N: Set of integers $\{1,2,\ldots,n\}$.

N_0: Set of integers $\{0,1,\ldots,n\}$.

R^m: Set of m-dimensional real column vectors, with coordinates generally indexed 1 through m. However, the coordinates of R^{n+1} are indexed by N_0.

u^i: ith unit vector in R^n, $i \in N$; $u = \sum_{i \in N} u^i$.

v^j: jth unit vector in R^{n+1}, $j \in N_0$; $v = \sum_{j \in N_0} v^j$.

R^m_+: Nonnegative orthant of R^m; $\{x \in R^m | x \geq 0\}$.

$C \cup D$, $C \cap D$, $C \sim D$: Union, intersection and set difference of the sets C and D.

$C + D$, $C - D$: Algebraic sum and difference of sets C and D in R^m;

$\{c + d | c \in C, d \in D\}$ and $\{c - d | c \in C, d \in D\}$ respectively.

λC: $\{\lambda c | c \in C\}$.

$\|x\|_2$: Euclidean norm of the vector $x \in R^m$; $(\sum_i x_i^2)^{1/2}$.

$\|x\|_\infty$: ℓ_∞ norm of $x \in R^m$; $\max_i |x_i|$.

$\|A\|_p$: The ℓ_p operator norm; $\max\{\|Ax\|_p \,|\, \|x\|_p = 1\}$ for $p = 2, \infty$ where A is a real $k \times m$ matrix.

B^m: Euclidean unit ball in R^m; $\{x \in R^m | \|x\|_2 \leq 1\}$.

$B(x,\rho)$: Ball with center x radius ρ; $\{x\} + \rho B^m$ if $x \in R^m$.

$B(C,\rho)$: $C + \rho B^m$ if $C \subseteq R^m$.

$\overline{C}$: Closure of C; $\cap \{B(C,\varepsilon) | \varepsilon > 0\}$.

int C: Interior of C; $\{x \in C | \exists\, \varepsilon > 0$ with $B(x,\varepsilon) \subseteq C\}$.

$\text{diam}_p C$: Diameter of C; $\sup\{\|x-y\|_p | x,y \in C\}$, $p = 2$ or ∞.

$\text{mesh}_p G$: Mesh of G; $\sup\{\text{diam}_p C | C \in G\}$ for $p = 2$ or ∞, where G is a family of subsets of R^m.