

**Texts and  
Monographs  
in Physics**

**W. Beiglböck  
M. Goldhaber  
E. H. Lieb  
W. Thirring**

*Series Editors*

Robert D. Richtmyer

# Principles of Advanced Mathematical Physics

Volume I



Springer-Verlag  
New York Heidelberg Berlin

**Robert D. Richtmyer**

Department of Physics and Astrophysics  
University of Colorado  
Boulder, Colorado 80309  
USA

*Editors:*

**Wolf Beiglböck**

Institut für Angewandte Mathematik  
Universität Heidelberg  
Im Neuenheimer Feld 5  
D-6900 Heidelberg 1  
Federal Republic of Germany

**Maurice Goldhaber**

Department of Physics  
Brookhaven National Laboratory  
Associated Universities, Inc.  
Upton, NY 11973  
USA

**Elliott H. Lieb**

Department of Physics  
Joseph Henry Laboratories  
Princeton University  
P O. Box 708  
Princeton, NJ 08540  
USA

**Walter Thirring**

Institut für Theoretische Physik  
der Universität Wien  
Boltzmanngasse 5  
A-1090 Wien  
Austria

**With 45 Figures**

ISBN-13: 978-3-642-46380-8

e-ISBN-13: 978-3-642-46378-5

DOI: 10.1007/978-3-642-46378-5

**Library of Congress Cataloging in Publication Data**

Richtmyer, Robert D.

Principles of advanced mathematical physics.  
(Texts and monographs in physics)

CONTENTS: v. 1. Hilbert and Banach spaces, distributions, operators, probability, applications to quantum mechanics, equations of evolution in physics.

Includes index.

1. Mathematical physics. I. Title.

QC20.R56 530.1'5 78-16494

All rights reserved.

No part of this book may be translated or reproduced in any form without written permission from Springer-Verlag.

© 1978 by Springer-Verlag New York Inc.

Softcover reprint of the hardcover 1st edition 1978

9 8 7 6 5 4 3 2 1

# Contents

	<b>Preface</b>	<b>xi</b>
<b>1</b>	<b>Hilbert Spaces</b>	<b>1</b>
	1.1 Review of pertinent facts about matrices and finite-dimensional spaces	1
	1.2 Linear spaces; normed linear spaces	3
	1.3 Hilbert space: axioms and elementary consequences	4
	1.4 Examples of Hilbert spaces	6
	1.5 Cardinal numbers; separability; dimension	8
	1.6 Orthonormal sequences	11
	1.7 Subspaces; the projection theorem	14
	1.8 Linear functionals; the Riesz–Fréchet representation theorem	16
	1.9 Strong and weak convergence	16
	1.10 Hilbert spaces of analytic functions	17
	1.11 Polarization	17
<b>2</b>	<b>Distributions; General Properties</b>	<b>19</b>
	2.1 Origin of the distribution concept	19
	2.2 Classes of test functions; functions of type $C_0^\infty$	21
	2.3 Notations for distributions; the bilinear form	22
	2.4 The formal definition; the continuity of the functionals	24
	2.5 Examples of distributions	26

2.6	Distributions as limits of sequences of functions; convergence of distributions	29
2.7	Differentiation and integration	31
2.8	Change of independent variable; symmetries	33
2.9	Restrictions, limitations, and warnings	35
2.10	Regularization	39
	Appendix: A discontinuous linear functional	40
<b>3</b>	<b>Local Properties of Distributions</b>	<b>43</b>
3.1	Quick review of open and closed sets in $\mathbb{R}^n$	43
3.2	Local properties defined	45
3.3	A theorem on open coverings	46
3.4	Theorems on test functions; partitions of unity	48
3.5	The main theorems on local properties	50
3.6	The support of a distribution	51
<b>4</b>	<b>Tempered Distributions and Fourier Transforms</b>	<b>52</b>
4.1	The space $\mathcal{S}$	52
4.2	Tempered distributions	53
4.3	Growth at infinity	54
4.4	Fourier transformation in $\mathcal{S}$	55
4.5	Fourier transforms of tempered distributions	56
4.6	The power spectrum	60
<b>5</b>	<b><math>L^2</math> Spaces</b>	<b>68</b>
5.1	Mean convergence; completeness of function systems	68
5.2	A physical example of approximation in the mean	73
5.3	The spaces $L^2(\mathbb{R}^n)$ and $L^2(\Omega)$	73
5.4	Multiplication in $L^2$ spaces	81
5.5	Integration in $L^2$ spaces; definite integrals	82
5.6	On vanishing at infinity I	85
5.7	Spaces of type $L^1, L^p, L^\infty$	86
5.8	Fourier transforms in $L^1$ ; Riemann–Lebesgue Lemma; Luzin’s theorem	89
5.9	Spaces of type $L^2_\sigma$	91
5.10	Fourier transforms and mollifiers in $L^2$ spaces	93
5.11	The Sobolev spaces; the space $W^1$	94
5.12	Boundary values in $W^1$ ; the subspace $W^1_0$	96
5.13	On vanishing at infinity II	97
<b>6</b>	<b>Some Problems Connected with the Laplacian</b>	<b>99</b>
6.1	The potential; Poisson’s equation	100
6.2	Convolutions	100

6.3	Proof of Poisson's equation	102
6.4	The classical potential-theory problems of Poisson, Dirichlet, Green, and Neumann	103
6.5	Schwartz's nuclear theorem; the direct product $f(x)g(y)$ .	108
6.6	The variational method for the eigenfunctions of the Laplacian	110
6.7	A compactness theorem for the Sobolev space $W^1$	113
6.8	Existence of the eigenfunctions	116
6.9	A problem from hydrodynamical stability; irrotational and solenoidal vector fields	118
6.10	The Cauchy–Riemann equations; harmonic distributions	123
<b>7</b>	<b>Linear Operators in a Hilbert Space</b>	<b>125</b>
7.1	Linear operators	125
7.2	Adjoins; self-adjoint and unitary operators	127
7.3	Examples in $l^2$	130
7.4	Integral operators in $L^2(a, b)$	130
7.5	Differential operators via distribution theory	131
7.6	Closed operators	135
7.7	The graph of an operator; range and nullspace	138
7.8	The radial momentum operators	139
7.9	Positive operators; numerical range	141
<b>8</b>	<b>Spectrum and Resolvent</b>	<b>143</b>
8.1	Definitions	143
8.2	Examples and exercises	144
8.3	Spectra of symmetric, self-adjoint, and unitary operators	147
8.4	Modification of the spectrum when an operator is extended	149
8.5	Analytic properties of the resolvent	151
8.6	Extension of a symmetric operator; deficiency indices; the Cayley transform; second definition of self-adjointness	153
<b>9</b>	<b>Spectral Decomposition of Self-Adjoint and Unitary Operators</b>	<b>158</b>
9.1	Spectral decompositions of a Hermitian matrix	158
9.2	Projectors in a Hilbert space $\mathfrak{H}$	160
9.3	Construction of the spectral projectors for a matrix	161
9.4	Connection with analytic functions	165
9.5	Functions and distributions as boundary values of analytic functions	167
9.6	The resolution of the identity for a self-adjoint operator	171
9.7	The properties of the operators $E_i$	173
9.8	The canonical representation of a self-adjoint operator	174
9.9	Modes of convergence of bounded operators; connection between the continuity properties of $E_i$ and the spectrum of $A$	176

9.10	Unitary operators; functions of operators; bounded observables; polar decomposition	181
	Appendix A: The properties of the operators $E_i$	184
	Appendix B: The canonical representations of a self-adjoint operator	187
<b>10</b>	<b>Ordinary Differential Operators</b>	<b>190</b>
10.1	Resolvent and spectral family for the operator $-id/dx$	190
10.2	Resolvent and spectral family for the operator $-(d/dx)^2$	191
10.3	The Fourier transform method	192
10.4	Regular Sturm–Liouville operator	194
10.5	Existence and uniqueness of the solution; the integral equation; the eigenfunctions	195
10.6	The resolvent; the Green’s function; completeness of the eigenfunctions	197
10.7	More general boundary conditions	198
10.8	Sturm–Liouville operator with one singular endpoint	199
10.9	The boundary condition at a singular endpoint	200
10.10	Regular singular point; method of Frobenius	203
10.11	Self-adjoint extension of $T$ in the limit-point case	205
10.12	The eigenfunction expansion	206
10.13	The limit-circle case	209
10.14	Case of two singular endpoints	210
10.15	Bessel’s equation	213
10.16	The nonrelativistic hydrogen-like atom	216
10.17	The relativistic hydrogen-like atom	218
<b>11</b>	<b>Some Partial Differential Operators of Quantum Mechanics</b>	<b>222</b>
11.1	Self-adjoint Laplacian in $\mathbb{R}^n$	222
11.2	Resolvent, spectrum, and spectral projectors	224
11.3	Schrödinger operators	
11.4	Perturbation of the spectrum; essential spectrum; absolutely continuous spectrum	228
11.5	Continuous spectrum in the sense of Hilbert; continuous and absolutely continuous subspaces	230
11.6	Dirac Hamiltonians	233
11.7	The Laplacian in a bounded region	238
<b>12</b>	<b>Compact, Hilbert–Schmidt, and Trace-Class Operators</b>	<b>241</b>
12.1	Some properties of matrices	241
12.2	Compact operators	242
12.3	Hilbert–Schmidt and trace-class operators	244
12.4	Hilbert–Schmidt integral operators	247
12.5	Operators with compact resolvent	248

<b>13</b>	<b>Probability; Measure</b>	<b>253</b>
	13.1 Univariate or one-dimensional probability distributions: cumulative probability; density	254
	13.2 Means and expectations	260
	13.3 Bivariate and multivariate distributions; nondecreasing functions of several variables	263
	13.4 The normal distributions	266
	13.5 The central limit theorem	269
	13.6 Sampling	273
	13.7 Marginal and conditional probabilities	276
	13.8 Simulation; the Monte Carlo Method	278
	13.9 Measures	281
	13.10 Measures as set functions	285
	13.11 Probability in Hilbert space; cylinder sets; Gaussian measures	291
	Appendix: Functions of Bounded Variation	295
<b>14</b>	<b>Probability and Operators in Quantum Mechanics</b>	<b>299</b>
	14.1 States of a system; observables	299
	14.2 Probabilities—a finite model	300
	14.3 Probabilities—the general case ( $\mathfrak{H}$ infinite-dimensional)	302
	14.4 Expectations; the domain of $A$	304
	14.5 The density matrix	306
	14.6 Algebras of bounded operators; canonical commutation relations	309
	14.7 Self-adjoint operator with a simple spectrum	312
	14.8 Spectral representation of $\mathfrak{H}$ for a self-adjoint operator with a simple spectrum	314
	14.9 Complete set of commuting observables	317
<b>15</b>	<b>Problems of Evolution; Banach Spaces</b>	<b>320</b>
	15.1 Initial-value problems in mechanics	320
	15.2 Initial-value problems of heat flow	321
	15.3 Well- and ill-posed problems	324
	15.4 The initial-value problem of wave motion	325
	15.5 The function space (state space) of an initial-value problem	326
	15.6 Completeness of the state space; Banach space	327
	15.7 Examples of Banach spaces	327
	15.8 Inequivalence of various Banach spaces	330
	15.9 Linear operators	331
	15.10 Linear functionals; the dual space	332
	15.11 Convergence of vectors and operators	332
	15.12 Inner product; Hilbert space	333
	15.13 Relativistic problems	333
	15.14 Seminorms	333

<b>16</b>	<b>Well-Posed Initial-Value Problems; Semigroups</b>	<b>335</b>
16.1	Banach-space formulation of an initial-value problem	335
16.2	Well-posed problem; generalized solutions	336
16.3	Wave motion	339
16.4	The Schrödinger equation	344
16.5	Maxwell's equations in empty space	347
16.6	Semigroups	350
16.7	The infinitesimal generator of a semigroup	351
16.8	The Hille–Yōsida theorem	354
16.9	Neutron transport in a slab; an application of the Hille–Yōsida theorem	355
16.10	Inhomogeneous problems	358
16.11	Problems in which the operator is time-dependent	362
<b>17</b>	<b>Nonlinear Problems; Fluid Dynamics</b>	<b>364</b>
17.1	Wave propagation	365
17.2	Fluid-dynamical conservation laws	366
17.3	Weak solutions	369
17.4	The jump conditions	370
17.5	Shocks and slip surfaces	372
17.6	Instability of negative shocks	373
17.7	Sound waves and characteristics in one dimension	375
17.8	Hyperbolic systems	377
17.9	Fluid-dynamical equations in characteristic form	378
17.10	Remarks on the initial-value problem	379
17.11	Flow of information along the characteristics in one dimension	382
17.12	Characteristics in several dimensions; the Cauchy–Kovalevski theorem	383
17.13	The Riemann problem and its generalizations	386
17.14	The spontaneous generation of shocks	388
17.15	Helmholtz and Taylor instabilities	390
17.16	A conjecture on piecewise analytic initial-value problems of fluid dynamics	393
17.17	Singularities of flows	393
	Appendix: The detached shock problem:	
17.A	The Problem	395
17.B	Ill-posedness of the problem	399
17.C	The power series method	401
17.D	Significance arithmetic	404
17.E	Analytic continuation	406
	<b>References</b>	<b>409</b>
	<b>Index</b>	<b>413</b>

# Preface: On the Nature of Mathematical Physics

Reasoning in mathematics and reasoning in physics have very different textures. Mathematics is held together by short-range forces that bind each step in a deduction directly to the preceding steps, whereas physics is held together by the much longer-range forces of analogy and intuition and all sorts of indirect supporting evidence. The comparison of physical science with cryptanalysis (“deciphering the secrets of nature,” etc.), though overworked, is apt. When one has solved a cipher and the message rings out loud and clear, one does not think of calling in a mathematician to provide a uniqueness proof, even though conceivably there might be a different solution, i.e., a different message. In physics, existence and uniqueness proofs are many decades behind current research (because of the inherent complexity of nature) and are often somewhat irrelevant, because they can be no more convincing than the hypotheses on which they are based, which in turn are matters of physics, while a large body of indirect evidence is often fully convincing. In mathematics, on the other hand, intuition and analogy are notoriously untrustworthy; although they often lead to useful conjectures, the conjectures never become part of the structure until proved. When one is proving a theorem in mathematics, one is not permitted to use any hypotheses except those present in the statement of the theorem.

A first consequence of this difference in texture concerns the attitude we must take toward some (or perhaps most) investigations in “applied mathematics,” at least when the mathematics is applied to physics. Namely, those investigations have to be regarded as pure mathematics and evaluated as such. For example, some of my mathematical colleagues have worked in recent years on the Hartree–Fock approximate method for determining the structures of many-electron atoms and ions. When the method was introduced, nearly fifty years ago, physicists did the best they could to justify it, using variational principles, intuition, and other techniques within the texture of physical reasoning. By now the method has long since become part of the established structure of physics. The mathematical theorems that can be proved now (mostly for two- and three-electron systems, hence of limited interest for physics), have to be regarded as mathematics. If they are good mathematics (and I believe they are), that is justification enough. If they are not, there is no basis for saying that the work is being done to help the physicists. In that sense, applied mathematics plays no role in today’s physics. In today’s division of labor, the task of the mathematician is to create mathematics, in whatever area, without being much concerned about how the mathematics is used; that should be decided in the future and by physics.

Specialization has, of course, gone too far, but even with less of it, it would be out of the question for the methods of contemporary mathematics to be transplanted all the way over into the area of contemporary physics and produce significant results. The differences are just too great. Today’s physicists know how to use mathematics; they know how to formulate problems, devise methods of solution, and perform long derivations and calculations, but they cannot create the mathematics. Experience has shown that the discovery and purification of abstract concepts and principles is peculiarly in the realm of mathematics. The division of labor is important and ought to be taken seriously.

There is no objection to a mathematician’s working in the areas that have come to be designated as applied mathematics, and if he can derive inspiration for his mathematics from the physical world, that is very much to the good, but the value to physics of the fruits of his labor will be determined by their quality as mathematics.

There is also no objection to a mathematician’s doing physics, provided he is qualified. The prime example was von Neumann—when he did physics, he talked, thought, and calculated like a physicist (but faster). He understood all branches of physics (including elementary particles as they were known then), and chemistry and astronomy, and he had a talent for introducing those and only those mathematical ideas that were relevant to the physics at hand. Anyone, regardless of professional affiliation, who can do physics one tenth that well should be encouraged to do it, but the objectives and methods are quite different from those of applied mathematics, whose purpose is to create mathematics.

Here are some quotations from Hardy's "A Mathematician's Apology":

1. "I said that a mathematician was a maker of patterns of ideas, and that beauty and seriousness were the criteria by which his patterns should be judged." (page 98)
2. "It is not possible to justify the life of any genuine professional mathematician on the ground of the 'utility' of his work." (page 119)
3. "One rather curious conclusion emerges, that pure mathematics is on the whole distinctly more useful than applied." (page 134)
4. "I hope I need not say that I am not trying to decry mathematical physics, a splendid subject with tremendous problems where the finest imaginations have run riot." (page 135)

Another consequence of the difference in texture concerns the word "rigor," which is badly misused by both mathematicians and physicists and possibly ought to be banished from our language. Physicists think mathematicians spend an inordinate amount of time making sure that all i's are dotted and all t's crossed, and the mathematicians shake their heads and wonder how those sloppy physicists ever get anything right. Both attitudes result from failure to recognize the methodological difference between the two disciplines. The situation becomes a little clearer when one teaches mathematics to physicists, for it turns out that although the physicists are not to be deterred from their accepted and successful ways of investigating the physical world, they demand rigor in mathematics. They want to know exactly what is true and what is false and exactly why (although they are eager to be told lots of additional things without proof), and they want to see lots of examples and counterexamples, in order to delineate the areas of relevance of the theorems.

In one branch of physics, quantum field theory, the difference in texture has almost disappeared, owing to the failure of the traditional methods. In 1900 Max Planck said "let's quantize the electromagnetic field," and he showed what wonderful things would happen if we could do it. Einstein showed more. In a certain measure, all modern physics is based on that suggestion, but the task has proved to be enormously more difficult than was supposed. In many attempts during the first half of this century, based on the intuitive methods that had been so successful in other parts of quantum mechanics, emission and absorption rates and line breadths were successfully calculated, but only by arbitrary suppression of infinities and inconsistencies, and for the most part in cases where the required result was already known from experiment and cruder theories. In the 1950s, various physicists began to take seriously the suppression of the infinities by the introduction of precise new axioms ("renormalization"), and a flood of exciting new results came out (Lamb shift, precise magnetic moments, etc.). Still, we do not yet have a water-tight theory, and each new attempt to overcome the difficulties of previous attempts has involved the introduction of more precise and more powerful mathematical tools. It now seems that intuitive

methods are just as untrustworthy in quantum field theory as in pure mathematics, and contemporary work in field theory has very much the same texture as pure mathematics; there is the feeling of “definition, lemma, proof, theorem, proof, etc.” if not the actual words. Presumably, when success finally comes, it will be through interplay between physical intuition and the newly found mathematical rigor.

The consequence for mathematical physics is an increased relevance of the careful study of operators, distributions, Banach algebras, functions of several complex variables, representations of noncompact groups, and so on.

People in other areas are usually unaware of the wide range of mathematics now used in physics. They assume that physicists are interested only in analysis and specifically the part of analysis appropriate for nineteenth-century physics, as set forth in Courant and Hilbert. Most books, including recent ones, on “mathematical methods for physicists,” and the like, contain no group theory, which has played an important role in physics since about 1925, and the authors give no indication they have ever heard of the mathematical principles and concepts basic to modern quantum mechanics, relativity, cosmology, scattering theory, quantum field theory, statistical mechanics, topological dynamical systems, and so on, to say nothing of the concepts and principles that have not yet found their way into physics, but are likely to do so in the near future and are likely to come from areas such as algebra, logic, set theory, and topology. No part of mathematics is devoid of potential interest for physics.

For our purpose, mathematical concepts and principles are more important than methods, and the main goal of courses in mathematical physics, in my opinion, is to explain the concepts and principles in such a way that one can see their relevance for physics. Here is an example:

*Manifolds in relativity:* In 1916 Karl Schwarzschild derived the static spherical solution of the Einstein field equations in the form now known by his name. His formula appeared to indicate some sort of singularity at a radius now called the Schwarzschild radius. There followed forty-four years of confusion about the “Schwarzschild singularity.” As time went by, it became gradually clear that Schwarzschild’s formula described only a part of the relevant physical space-time, and in 1960 Martin Kruskal gave a description of the geodesically complete manifold of which Schwarzschild’s formula determined a part. It was then seen that although certain interesting phenomena are associated with the Schwarzschild radius, there is no singularity there. Relativists now take the attitude that by a solution of the Einstein equations one has to understand not just a formula for a line element  $ds^2 = \dots$ , but rather a complete manifold, and that the global topology of the manifold may be of cosmological significance. The introduction of the geometric notion of *manifold* into relativity is a prime example of mathematical physics. The theory of manifolds is set forth in Volume II.

An earlier example was the introduction of abstract Hilbert space theory into quantum mechanics, mainly by von Neumann, which made it possible to construct a solid theory on the basis of the powerful intuitive ideas of

Dirac and other physicists. No less important was the introduction of groups and group representations, mainly by Wigner and Weyl.

A recent example is the introduction by Ruelle and Takens of ideas from the topological theory of differentiable dynamical systems into the study of the onset of turbulence. These ideas are likely to play a role in other parts of physics, where nonlinear differential equations appear.

The basic mathematics of physics belongs in physics courses. The proper formulation of boundary-value problems, asymptotic expansions, consequences of symmetries, and so on are all matters of physics. Although the ideas are further clarified and analyzed in mathematical physics courses, their first introduction should appear as part of the physics. A physics instructor ought never to say to his students, "just how those things work will be explained in your mathematics courses." Physics and mathematics cannot be separated in that way, and it is not the purpose of courses in mathematical physics to relieve physics teachers of the responsibility of explaining their subject. In practice, however, the best physics courses cannot be adequate on all topics. For example, most books on quantum mechanics are hopelessly unclear about Hilbert spaces and operators, and students need to learn about those things after they have first studied quantum mechanics as an intuitive subject. It is not just a question of "rigor." Whether a given symmetric operator has self-adjoint extensions and, if so, how many different ones, is a matter of physics, because the self-adjoint operators are the observables. The probabilistic interpretation of the spectral family of a self-adjoint operator gives the physical interpretation of the observable even for states that are not in the domain of the operator, and so on. I interpret mathematical physics so as to include the explanation of these things.

Most good ideas turn out to be simple ones, and I believe it is important that they be so presented, without unnecessary ramification of other ideas. In my view, for example, distribution theory should be based (rigorously, of course) on the Riemann integral and advanced calculus, and  $L^2$  spaces and the theory of differential operators should be based on distribution theory. The students can learn about measure theory and topological vector spaces later. It has been my intention in these two volumes to present the fundamental ideas in the most down-to-earth fashion possible. At the same time, I have not hesitated to introduce further ideas that have independent interest, for example transfinite cardinals in the chapter on Hilbert spaces.

*Boulder, December 1978*

Robert D. Richtmyer