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The Theory of Max-Min

and its Application to Weapons Allocation Problems

John M. Danskin

With 6 Figures

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This book is dedicated to my father

JOHN M. DANSKIN, SR.

who made it possible for

me to become a

mathematician

Preface

Max-Min problems are two-step allocation problems in which one side must make his move knowing that the other side will then learn what the move is and optimally counter. They are fundamental in particular to military weapons-selection problems involving large systems such as Minuteman or Polaris, where the systems in the mix are so large that they cannot be concealed from an opponent. One must then expect the opponent to determine on an optimal mixture of, in the case mentioned above, anti-Minuteman and anti-submarine effort.

The author's first introduction to a problem of Max-Min type occurred at The RAND Corporation about 1951. One side allocates anti-missile defenses to various cities. The other side observes this allocation and then allocates missiles to those cities. If $F(x, y)$ denotes the total residual value of the cities after the attack, with x denoting the defender's strategy and y the attacker's, the problem is then to find

$$\operatorname*{Max}_{x} \operatorname*{Min}_{y} F(x, y) = \operatorname*{Max}_{x} \left[\operatorname*{Min}_{y} F(x, y) \right].$$

If it happens that

$$\operatorname*{Max}_{x} \operatorname*{Min}_{y} F(x, y) = \operatorname*{Min}_{y} \operatorname*{Max}_{x} F(x, y),$$

the problem is a standard game-theory problem with a pure-strategy solution. If however

$$\operatorname*{Max}_{x} \operatorname*{Min}_{y} F(x, y) < \operatorname*{Min}_{y} \operatorname*{Max}_{x} F(x, y),$$

i.e., the order of the choices of x and y is essential, then standard game theory fails. The concept of mixed strategy has no meaning: the x-player knows that his strategy will be observed by his opponent, and the y-player knows x when he acts and needs simply to minimize. Thus the problem needs a separate treatment. That is the object of this book.

It is natural to begin by studying the nature of the function

$$\varphi(x) = \operatorname*{Min}_{y} F(x, y) \qquad (*)$$

which is to be maximized. The principal difficulty, illustrated on an example (the "seesaw") in Chapter I, is that $\varphi(x)$ is not in general differentiable in the usual sense, even when $F(x, y)$ is quite smooth. This is closely connected with non-uniqueness in the set $Y(x)$ of values of y

yielding the minimum in (*). There however is under general conditions on the x- and y-spaces a directional derivative in every direction, given by a formula involving the "answering set" $Y(x)$.* With this result in hand it was then possible to develop a calculus complete with a law of the mean and a Lagrange multiplier theorem for one side condition, but complicated by the lack of a chain rule. Later J. BRAM found a Lagrange multiplier theorem for several side conditions analogous to the well-known Kuhn-Tucker theorem for a simple maximum of a smooth function.

It was now possible to treat a number of problems of Max-Min type. The RAND problem is treated in section 4 of Chapter IV. A problem on the optimal mixture of weapons systems (e.g., Polaris versus Minuteman) was treated for its mathematical interest and then became the theoretical basis of a study on weapons mixtures of the Institute of Naval Studies, the Cambridge, Massachusetts branch of the Center for Naval Analyses. Practical questions led to three-stage problems of Max-Min-Max type. Under some conditions explicit solutions of problems of Max-Min-Max type can be obtained. In the general case information concerning the solution seems to depend on the stability properties of the solutions to the "inside" problems; in the case of "strong forward stability" the solutions become trivial.

By now the material had grown to book-length. The writing was supported at the Center for Naval Analyses, to whose Director, Dr. FRANK BOTHWELL, I am grateful for the stimulus to write the book and for the time to do it. I wish to express also my thanks to Dr. JOSEPH BRAM and Mrs. SIDNIE FEIT, who read and criticized previous versions of portions of the book; Dr. BRAM has been kind enough to allow me the use of the material in the Appendix. There is finally Mrs. JANE LYNDE of the Institute of Naval Studies in Cambridge, Massachusetts, who typed the many versions of this book. The author is also very grateful to Captain DANIEL J. MURPHY, U.S.N. of the Office of the Chief of Naval Operations, whose interest and support greatly facilitated the writing of this book.

JOHN M. DANSKIN

* After this book was finished, the author's attention was called to a somewhat similar theorem on the derivative of the value of a game, due to OLIVER GROSS and referred to in a paper [7] by HARLAN MILLS. The Gross theorem referred to matrix games and continuous games over the square. See also the addendum to [8].

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