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Geometry of Continued Fractions

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Preface

Continued fractions appear in many different branches of mathematics: the theory of Diophantine approximations, algebraic number theory, coding theory, toric geometry, dynamical systems, ergodic theory, topology, etc. One of the metamathematical explanations of this phenomenon is based on an interesting structure of the set of real numbers endowed with two operations: addition $a + b$ and inversion $1/b$. This structure appeared for the first time in the Euclidean algorithm, which was known several thousand years ago. Similarly to the structures of fields and rings (with operations of addition $a + b$ and multiplication $a * b$), structures with addition and inversion can be found in many branches of mathematics. That is the reason why continued fractions can be encountered far away from number theory. In particular, continued fractions have a geometric interpretation in terms of integer geometry, which we place as a cornerstone for this book.

The main goal of the first part of the book is to explore geometric ideas behind regular continued fractions. On the one hand, we present geometrical interpretation of classical theorems, such as the Gauss–Kuzmin theorem on the distribution of elements of continued fractions, Lagrange's theorem on the periodicity of continued fractions, and the algorithm of Gaussian reduction. On the other hand, we present some recent results related to toric geometry and the first steps of integer trigonometry of lattices. The first part is rather elementary and will be interesting for both students in mathematics and researchers. This part is a result of a series of lecture courses at the Graz University of Technology (Austria). The material is appropriate for master's and doctoral students who already have basic knowledge of linear algebra, algebraic number theory, and measure theory. Several chapters demand certain experience in differential and algebraic geometry. Nevertheless, I believe that it is possible for strong bachelor's students as well to understand this material.

In the second part of the book we study an integer geometric generalization of continued fractions to the multidimensional case. Such a generalization was first considered by F. Klein in 1895. Later, this subject was almost completely abandoned due to the computational complexity of the structure involved in the calculation of the generalized continued fractions. The interest in Klein's generalization was revived by V.I. Arnold approximately one hundred years after its invention, when computers became strong enough to overcome the computational complexity. After a brief introduction to multidimensional integer geometry, we study essentially new questions for the multidimensional cases and questions arising as extensions of the classical ones (such as Lagrange's theorem and Gauss–Kuzmin statistics). This part is an exposition of recent results in this area. We emphasize that the majority of examples and even certain statements of this part are on two-dimensional continued fractions. The situation in higher dimensions is more technical and less studied, and in many cases we formulate the corresponding problems and conjectures. The second part is intended mostly for researchers in the fields of algebraic number theory, Diophantine equations and approximations, and algebraic geometry. Several chapters of this part can be added to a course for master's or doctoral students.

Finally, I should mention many other interesting generalizations of continued fractions, coming from algorithmic, dynamical, and approximation properties of continued fractions. These generalizations are all distinct in higher dimensions. We briefly describe the most famous of them in Chap. 23.

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