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Deformations of Surface Singularities

Springer

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Mathematics Subject Classification (2010):
14B05, 14B07, 14B12, 32B10, 32S05, 32S25, 32S30, 53D05, 53D10, 57M27

Library of Congress Control Number: 2013941047

ISSN 1217-4696
ISBN 978-3-642-39130-9 Springer Berlin Heidelberg New York
ISBN 978-963-9453-16-6 János Bolyai Mathematical Society, Budapest

Springer is a part of Springer Science+Business Media
springer.com

Cover design: WMXDesign GmbH, Heidelberg

Printed on acid-free paper 44/3142/db – 5 4 3 2 1 0

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Preface

The present publication contains a special collection of research and review articles on *deformations of surface singularities,* that put together serve as an introductory survey of results and methods of the theory, as well as open problems, important examples and connections to other areas of mathematics, such as the theory of Stein fillings and symplectic geometry.

We envision this volume as a guide for all those already doing or wishing to do research in this area, and thus it is intended to be especially useful for PhD students.

1. A Short Introduction in Deformation Theory

Deformation theory appeared as the investigation of how complex structures may vary on a fixed compact, *smooth* manifold. In his famous paper "Theorie der abelschen Funktionen" [16], published in 1857, Riemann already mentioned the $3g - 3$ moduli determining the complex structure of an algebraic curve ('Riemann surface') of genus $g \geq 2$.

Looking for an analogous description in higher dimensions, Kodaira and Spencer started developing the machinery of what is called deformation theory today [12]. Because of the fact that (beginning in dimension two) a good moduli space does not always exist, they used a modified weaker concept: the versal (or semi-universal) deformation $f : X \to S$ of a manifold $X_0 = f^{-1}(0)$ $(0 \in S)$. This space parametrizes all possible deformations (but even the minimal one no longer provides a one-to-one correspondence between fibers and complex structures).

Kodaira and Spencer showed the existence of the mini-versal deformation space ('universal versal deformation space'), under a certain cohomological vanishing. Later Kuranishi [14] completed their work by allowing singular base spaces and eliminating the cohomological assumption.

In a similar manner, we may regard deformations of *germs of analytic spaces* (or equivalently, deformations of local, analytic $\mathbb{C}$-algebras) that are not necessarily smooth: that are 'singularities'. In this case one defines the possible deformation spaces, deformation functors, and the notion of mini-versal deformations as well. It is not hard to see that the mini-versal deformation (if it exists) is uniquely determined; moreover, in the case of complete intersections the base space is smooth. However, in general, the base space and the structure of the versal family might become extremely complicated.

The existence of the analytic mini-versal space was established (under some assumptions) by Schlessinger [18]. For normal surface singularities the existence-problem was completely solved by Tjurina, and the most general case by Grauert [6] in 1972.

Schlessinger's method leads to the construction of the versal deformations, and useful criterion to verify versality. The reader might consult for more information Artin's Lecture notes [4], Palamodov's large introduction [15], or J. Stevens Thesis [21] and his recent Springer LNM–book [23].

Deformation theory of *normal surface singularities* in the last decades witnessed an extraordinary development in spite of being one of the most difficult subjects of singularity theory, as it is based on hard machinery from algebraic geometry, sheaf-cohomology and algebraic topology.

As the singularity is encoded in the construction of the mini-versal deformation space (and/or its base-space), this space contains important information about the given germ and is a crucial source of numerical invariants. Its most important two ingredients are the *tangent space* and *obstruction space,* which became the subject of intense mathematical study in the last years. Let us list some key examples.

Riemenschneider [17] and Arndt [3] initiated the description of *cyclic quotient singularities,* particularly of their tangent space. Its construction, as well as of the obstruction space, was completed by Christophersen [5] and Behnke. We emphasize that already in this particular case the deformation space is not smooth, and it contains many irreducible components. The fact that in the above construction one indeed obtains all the components of the deformation space was verified by J. Stevens [22] based on the article of Kollár and Shepherd–Barron [13].

The next steps were again seriously obstructed: it proved to be very hard to generalize the results valid for cyclic quotient singularities. Nevertheless, D. van Straten and T. de Jong using new ideas obtained positive results in the direction of (the still open) Kollár Conjecture (targeting the description of the base space of the mini-versal deformation) in case of *rational quadruple singularities* [7] and *minimal rational singularities* [8]. Moreover, in a series of articles they developed a whole 'deformation theory of non-isolated singularities' [9, 10]. Their theory was successfully used for many families of normal surface singularities, e.g. for 'sandwiched' and rational singularities, applied to their projection in $(\mathbb{C}^3, 0)$ [11]. Their article [11] describes the smoothings (i.e. those deformations which provide smooth deformation fibers) of sandwiched singularities,—an important family with testing characteristics for any new theory, introduced by M. Spivakovsky [19, 20].

Recently, smoothings of rational and sandwiched singularities became a focus of interest in Contact Geometry and Stein/symplectic fillings as well: they provide the most important models of the theory. Indeed, local Milnor fibers are particular Stein fillings of the corresponding singularity links.

Simultaneously, Teissier (and his school), Laufer and Wahl developed the theory of ('very weak', 'weak', 'strong') simultaneous resolutions, where deformation and resolution theory are combined (see e.g. [24]). This led to the development of *equisingularity theory* and its connections with commutative algebra (integral closures).

There is another class of singularities which is in the mainstream of the deformation research: those provided by toric geometry. Toric geometry is that part of algebraic geometry which identifies its object by combinatorial construction (etc. by integral polyhedrons, or rational fans), and targets the computation of all topological/algebraic/sheaf-theoretical invariants via combinatorics. Their deformation theory was developed by K. Altmann, (see e.g. [1], or [2]).

2. The 'Deformation Theory Conference' at Budapest

In the period 10–12 October, 2008, the Alfréd Rényi Institute of Mathematics (Hungarian Academy of Sciences in Budapest, Hungary), organized a meeting titled *Deformation of Surfaces* (organizers: A. Némethi and Á. Szilárd.)

At the meeting experts of this area—Klaus Altmann, Jan Christophersen, Helmut Hamm, Theo de Jong, Monique Lejeune-Jalabert, Mark Spivakovsky, Jan Stevens, Duco van Straten—gave lectures explaining the classical theory and constructions complemented with presentation of recent developments and open questions. A special emphasis was put on key classes of singularities such as *rational, sandwiched* and *minimal rational.*

Moreover, the local theory of surface singularities was related with the theory of affine surfaces and their deformations in the talk of M. Zaidenberg. (For a list of guiding open problems of Zaidenberg see [25].)

The talks given were:

D. van Straten: *Introduction (How should one think about deformation theory?).*

K. Altmann: *Introduction to the deformation theory of toric singularities.*

K. Altmann: *The smoothings of certain toric singularities described by quivers.*

J. Stevens: *Versal deformation of cyclic quotient singularities.*

J. Christophersen: *Deformations of Stanley-Reisner surfaces.*

M. Lejeune-Jalabert: *Integral closure of ideals and equisingularity.*

M. Spivakovsky: *Equisingular deformations of sandwiched surface singularities.*

T. de Jong & D. van Straten: *Deformation of minimal and sandwiched singularities I. and II.*

J. Stevens: *Open problems, interesting questions regarding deformations of surface singularities.*

H. A. Hamm: *Equisingularity of curves on surfaces.*

M. Zaidenberg: *Deformations of acyclic surfaces and of* $\mathbb{C}^*$*-actions.*

The meeting was an absolute success with 34 foreign participants, including a large number of PhD students. It was the great interest shown by the audience because of which the idea of a volume collecting *research and review articles* on deformations of surface singularities, that put together would serve as a comprehensive survey of the key results and methods in this area known today, as well as open problems, important examples and connections to other areas of mathematics was conceived.

Thus the aim of the present volume is to collect material that will help mathematicians already working or wishing to work in this area to deepen their insight and eliminate the technical barriers in this learning process. This also is supported by review articles providing some global picture and abundance of examples.

Additionally, we introduce some material which emphasizes the newly found relationship with the theory of Stein fillings and Symplectic geometry (work of Eliashberg, Ono, Ohta, Etnyre, Lisca, Stipsicz). This links two main theories of mathematics: low dimensional topology with algebraic geometry.

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