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Editors

Monomial Ideals, Computations and Applications

 Springer

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Introduction

Monomial ideals and algebras are among the simplest structures in commutative algebra and the main objects in combinatorial commutative algebra. One can highlight several characteristics of these objects that make them of special interest.

One of the main points related to monomial ideals in commutative algebra and algebraic geometry is that some problems on general polynomial ideals can be reduced to the special case of monomial ideals through the theory of Gröbner basis. For example, the dimension of an affine (or a projective) algebraic variety defined by an arbitrary polynomial ideal I coincides with the dimension of the variety defined by a monomial ideal associated with I , the *initial ideal* (or *leading term ideal*) of I with respect to a term ordering. On the other hand, the dimension of the variety defined by a monomial ideal can be read off from its generators easily. This example illustrates how a problem in commutative algebra and algebraic geometry (the computation of the dimension of an arbitrary algebraic variety) can be reduced to an easier problem on monomial ideals. From a computational point of view, the combinatorial nature of monomial ideals makes them particularly well suited for the development of algorithms to work with and then generate algorithms focused on more general structures. The most important computer algebra systems specialized in polynomial computations, like CoCoA-5 [39], *Macaulay 2* [47] or Singular [80] which are, nowadays, a fundamental tool for the working commutative algebraist or algebraic geometer, make a heavy use of this fact. Besides its use for computational issues, the theory of Gröbner basis turned out to be a powerful tool for solving theoretical problems in commutative algebra and algebraic geometry.

Another fundamental aspect of monomial ideals is that their combinatorial structure connects them to other combinatorial objects and allows the resolution of problems at each side of this correspondence with the techniques of each of the respective areas. The work of Melvin Hochster and Richard Stanley, in particular [180], gave an important impetus to the development of these ideas, and the classical book of Stanley, [182], reflects how the areas of combinatorics and commutative algebra clearly interact. This led to the creation of a new and very active field, *combinatorial commutative algebra*.

The last decade has witnessed a plethora of different ideas, results and approaches to monomial ideals and their role in commutative algebra and its applications. This volume presents three different and representative topics on this area. They are written by leading experts in the field and give an account of the state of the art of the work on monomial ideals. The topics and the way to present them have been chosen so that the peculiarities of research in monomial ideals are highlighted together with the richness of approaches that these structures allow. The order of the chapters intend to guide the reader in the discovery of the variety of topics that arise in research on monomial ideals. The starting point of the book is in commutative algebra, the natural context in which monomial ideals are found. In the first part of this monograph, we can see how we can take advantage of the combinatorial nature of this type of ideals and use powerful tools of commutative algebra. From this conjunction, one obtains a first glance on the particular techniques one can use working with monomial ideals. The second part of the book shows how monomial ideals techniques share a common nature with certain combinatorial objects and tools. In this common area where monomial ideals lie, there are algebraic techniques that can be used for combinatorial problems and vice versa. Finally, in the third part one can see that the constructive character of the tools used when working with monomial ideals can also be used in topics of a more general and abstract nature in commutative algebra. After the journey through the three parts of this volume, the reader can have a neat impression on the essence of the work with monomial ideals by studying up-to-date research on three different contexts. The fact that each chapter includes a computer tutorial contributes to highlight one of the peculiarities of monomials ideals, that they are particularly well suited for producing algorithms and data types to study algebraic and combinatorial topics.

We now discuss the topics and authors of the three parts of this volume.

The first chapter of the book, by Jürgen Herzog, gives a survey on Stanley decompositions and their relation to the depth of a module and discusses the conjecture by Richard Stanley regarding this relationship. The conjecture remains wide open and research around its solution has provided a collection of concepts and results that involved many different authors. Jürgen Herzog has been very actively involved in this line of research and he gives in this chapter a first-hand survey where the conjecture is discussed in all its different aspects. The tutorial of this part, written by Anna M. Bigatti and Emanuela De Negri in the second chapter, uses the computer algebra system CoCoA-5 [39] to compute Stanley decompositions and allows the reader to explore the computational aspects of this topic.

The third chapter, by Adam Van Tuyl, features the basic properties of edge and cover ideals and introduces some current research themes. This topic has recently received much attention, producing interesting connections and beautiful results in the intersection of combinatorics and commutative algebra. The tutorial of this part, also written by Adam Van Tuyl in the fourth chapter, includes an introduction to the package `EdgeIdeals` [69] distributed with the computer algebra system *Macaulay 2* [47]. It also includes a list of exercises that can be solved, most of them, using the package `EdgeIdeals` and that illustrate the concepts and results introduced in this part.

Finally, the last part of the book gives an overview of some results that have been developed in recent years about the structure of local cohomology modules supported on a monomial ideal. This part is written by Josep Àlvarez Montaner in the fifth chapter where the interplay of multi-graded commutative algebra, combinatorics and D-modules theory is highlighted, providing different points of view on this subject that led to fundamental improvements in our understanding of these notions. The tutorial accompanying this part, written by Josep Àlvarez Montaner and Oscar Fernández-Ramos in the last chapter, proposes several examples and exercises to perform computations on local cohomology modules supported on monomial ideals. As for the second part of this volume, this tutorial uses the computer algebra system *Macaulay 2* [47], and the package `EdgeIdeals`, already featured in the fourth chapter, is used in some of the exercises.

This book originated from a series of lectures given by the authors at the conference *MONICA: MONomial Ideals, Computations and Applications* held at CIEM, Castro Urdiales (Cantabria, Spain) in July 2011. The MONICA meeting was organized by the editors of this volume and included, together with the three lecture series and their tutorials, some contributed talks by the participants, with a wide range of topics around monomial ideals and algebras. We want to thank all the participants and, in particular, the invited speakers and computer tutorial authors for the high quality of their contributions, showing the variety and excellence of research on monomial ideals and algebras. Last but not least, we finally want to thank i-math, CIEM, the city of Castro Urdiales, the University of Valladolid and the Spanish government (*Ministerio de Ciencia e Innovación*, grants MTM2010-20279-C02-02 and MTM2009-13842-C02-01) for providing financial support for the MONICA conference. We hope that this volume contributes to cover the lack of survey bibliography in an area in which such intense research is done in many directions.

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