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Tadahito Harima • Toshiaki Maeno
Hideaki Morita • Yasuhide Numata
Akihito Wachi • Junzo Watanabe

The Lefschetz Properties

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Tadahito Harima
Department of Mathematics Education
Niigata University
Nishi-Ku, Niigata, Japan

Toshiaki Maeno
Department of Mathematics
Meijo University
Nagoya, Japan

Hideaki Morita
Muroran Institute of Technology
Muroran, Japan

Yasuhide Numata
Department of Mathematical Sciences
Shinshu University
Matsumoto, Japan

Akihito Wachi
Department of Mathematics
Hokkaido University of Education
Kushiro, Japan

Junzo Watanabe
Department of Mathematics
Tokai University
Hiratsuka, Japan

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Introduction and Historical Note

An Artinian local ring is geometrically a point with multiplicity. The coordinate ring for such a point is a quotient of a regular local ring by an ideal of finite colength. Those ideals are, in contrast to ideals of low codimension, very complicated. It could be said that the study of Artinian rings is at the outset made difficult by their apparent simplicity and therefore lack of obvious invariants. Nonetheless, and precisely for this reason, there are good grounds to be interested in Artinian rings.

For example, (1) many problems of local rings often "reduce" to problems of Artinian rings, (2) standard graded Artinian Gorenstein algebras are parameterized by the n-ary r-ics, which were central objects of study during the early development of abstract algebra in the nineteenth century, their classification was the main problem of classical invariant theory, (3) a strong parallelism can be observed between Artinian rings and finite posets and this aids understanding of both, (4) there is no harm in looking at graded Artinian rings as cohomology rings of some (probably nonexistent) algebraic varieties, and if we do so, it gives us a new way of looking at Artinian rings, and (5) it helps us understand the Schur–Weyl duality which plays a very basic role in representation theory. All commutative algebraists would most likely agree with us if we said that the best way to understand the Schur–Weyl duality was through commutative algebra. (The interested reader may wish to see [103].)

The viewpoint (1) above is traditional and well understood. What we try to do in this book is to emphasize less known aspects of the theory of Artinian rings encompassed by (2)–(5).

A major theorem in the theory of finite sets is Dilworth's theorem, which states that the maximum size of antichains in a poset is equal to the minimum number of chains into which the poset can be disjointly decomposed. This common number is called the Dilworth number of the poset. This result is a typical minimax theorem and it is a powerful tool to determine the number for a poset. In fact, if one finds a disjoint chain decomposition and an antichain whose size is equal to the number of the chains, then this number is the Dilworth number. This idea works practically unchanged for Artinian rings. A minimal generating set of an ideal in an Artinian local ring is almost exactly an antichain in a poset. The counterpart of a chain decomposition of a poset is the Jordan block decomposition of a linear form in the regular representation of a local ring. With this interpretation, we may attach new meanings to Artinian rings and moreover provide a new method for proving Sperner type theorems in the theory of Artinian local rings. In his pioneering work [136], Stanley showed that the hard Lefschetz theorem can be used to prove that certain posets have the Sperner property.

The investigation of the Lefschetz properties of Artinian local rings was first suggested by one of the authors in the Japan–US joint seminar on combinatorics and commutative algebra held in Kyoto, Japan in the summer 1985, where R. Stanley gave an expository lecture, among other things, on $\mathfrak{sl}_2$-posets and J. Watanabe announced the results of [146] on the Sperner number of Artinian rings. However, for the first 10–15 years few results were obtained in this direction; there were many natural problems but they seemed too difficult to deal with.

The first breakthrough was made by J. Migliore and U. Nagel in [55], and after that many results were obtained and many connections to other areas of mathematics began emerging. These results are now known as Lefschetz properties of graded Artinian rings.

By now there are many results and applications on or of the Lefschetz properties and it would be a demanding task for any authors of a book to include all major results on the Lefschetz properties. In this monograph, the authors intend to concentrate on presenting the results from what has become a vast literature with which they are most familiar, many of which were discovered by the authors themselves in the last 10 years. Even this selection seems more than enough to show the diversity of applications of the Lefschetz properties. The authors have tried to put the emphasis on the diversity of applications and not on the completeness of the results presented here.

There are many topics on the Lefschetz properties that are being developed and investigated. For example, very recently the Lefschetz properties have been found to have surprising new connections with enumerative combinatorics (with plane partitions and lattice path enumeration, in particular). This was discovered algebraically by D. Cook II and U. Nagel [21, 22, 26] and independently by J. Li and F. Zanello [80] and more importantly was then explained combinatorically by Chen, Guo, Jin and Liu [15]. A few of their results are collected in the Appendix without proof. There is also a surprising relation to Laplace equations discovered by E. Mezzetti, M. -R. Miro–Róig and G. Ottaviani [8, 100]. Regretfully these are not included in this book.

There is a very good survey paper by Migliore and Nagel “A Tour of the Weak and Strong Lefschetz Properties” [101]. It is amazing that there is little overlapping between the topics dealt with in their survey and those in this monograph. The fact is an evidence for the diversity of the Lefschetz properties.

In Chap. 1 of this book we have collected some basic definitions and theorems of the Sperner theory of finite sets as are found in the books [2, 9, 33] and [34]. Particularly the treatment of Dilworth’s Theorem and the Sperner theory using the language of “set systems” as in [9] is of help to a newcomer to this area. The purpose of this section is to introduce the reader to the theory. Particularly important for us are the four examples of posets with rank function that we introduce here and are companions throughout the book. They have a direct translation in terms of Artinian rings.

In Chap. 2 we provide basic definitions and theorems of Artinian local rings with emphasis on Gorenstein algebras. Except for a few cases we restrict ourselves to Artinian rings and do not treat commutative rings in general, since Artinian Gorenstein algebras can be understood without knowledge of commutative algebra as a whole. This section is intended for those readers who wish to acquire knowledge of Artinian Gorenstein algebras as quickly as possible.

In Chaps. 3–6 we present recent results that were obtained and developed by the authors over the last 10 years.

The motivation in proving these results was a desire to prove that Artinian complete intersections have the strong Lefschetz property over a field of characteristic zero. We show some special cases where this holds. In Chap. 7 we relate the Lefschetz property of Artinian Gorenstein algebras with the hard Lefschetz theorem in algebraic geometry and show some ring-theoretic methods for proving certain cohomology rings have the strong Lefschetz property. In Chaps. 8 and 9 we show that the Lefschetz elements have a special meaning in representation theory.

$\mathfrak{m}$-Full Ideals and Dilworth Number of Artinian Rings

The number of generators of ideals in a ring has been one of the central topics in the theory of commutative rings. In fact some theorems on the number of generators have fundamental importance in the theory of commutative rings, for example, the Hilbert basis theorem, Krull's principal ideal theorem, and Serre's theorem which states that height two Gorenstein ideals are complete intersections. More results can be found in Sally's book [122].

In 1983, D. Rees raised the following problem: for which ideals I (say, in a local ring R with maximal ideal $\mathfrak{m}$), is it true that $\mu(I) \geq \mu(J)$ for any $J \supset I$. (Here μ denotes the number of generators.) To approach this problem, one has to ask oneself: what is the number

$$\max\{\, \mu(J) \mid J \supset I \,\}, \tag{1}$$

for a given I, where J runs over all ideals containing I. Set $A = R/\mathfrak{m}I$. Then this number is equal to

$$\max\{\, \mu(\mathfrak{a}) \mid \mathfrak{a} \subset A \,\}. \tag{2}$$

If this is finite, then it follows that Krull dim $R/I \leq 1$. Let us confine ourselves to the case the Krull dimension is zero or in other words, R/I is Artinian. If R is a polynomial ring over a field and I can be generated by monomials, then finding the number (1) is a combinatorial problem. This can be explained as follows.

Let $R = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring over a field K and let $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$ be the maximal ideal. Further, let S be the set of all monomials in R. Then S is a set partially ordered by divisibility. Let I be an ideal in R generated by monomials such that R/I is Artinian. Consider the set

$$P := S \setminus \mathfrak{m}I.$$

The set P, being a subset of S, is a poset. Notice that P is a vector space basis for $R/\mathfrak{m}I$. If we want to determine the number (2) for $A = R/\mathfrak{m}I$, we need to find a set of monomials in P of maximum size whose elements are totally disordered. The maximum number of elements in antichains (totally disordered sets) in a poset is known as the Dilworth number. The name came from the theorem of Dilworth, as mentioned earlier.

The definition of "Dilworth number" can be extended to all commutative rings of Krull dimension at most one, but the case of Artinian local rings is essential. The investigation made in [146] was motivated by a desire to determine the numbers (1) and (2) for $\mathfrak{m}$-primary ideals and Artinian local rings in general.

To answer Rees's question let $y \in R$ be any non-unit element and let J be any ideal of R which contains I. Then it is not difficult to prove that

$$\mu(J) \leq \text{length}(R/(\mathfrak{m}I + yR)).$$

(See Proposition 2.30.) For $y \in \mathfrak{m}$, the numerical value $\nu_y(I)$ for the ideal $I \subset R$ is defined by

$$\nu_y(I) = \text{length}(R/(\mathfrak{m}I + yR)).$$

Then obviously we have

$$I_1 \subset I_2 \implies \nu_y(I_1) \geq \nu_y(I_2).$$

Thus, for a given ideal I, if there exists an element $y \in \mathfrak{m}$ such that

$$\mu(I) = \nu_y(I), \tag{3}$$

then I has the property which Rees's question asks for. It is Rees himself who defined the notion of an $\mathfrak{m}$-full ideal. Namely, I is **$\mathfrak{m}$-full** if there exists an element y such that $\mathfrak{m}I : y = I$. This is equivalent to the claim that the equality (3) $\mu(I) = \nu_y(I)$ holds. Hence, we have the proposition stating that an $\mathfrak{m}$-primary $\mathfrak{m}$-full ideal has the Rees property. (See Proposition 2.55.)

It is easy to see that the intersection of two $\mathfrak{m}$-primary $\mathfrak{m}$-full ideals is $\mathfrak{m}$-full. Hence an $\mathfrak{m}$-primary ideal has an $\mathfrak{m}$-full closure. Given an $\mathfrak{m}$-primary ideal I, it seems natural to hope that, for any $\mathfrak{m}$-primary ideal I, we have

$$\mu(I^*) = \max \{ \mu(J) \mid J \supset I \},$$

where I^* is the $\mathfrak{m}$-full closure of I. This equality seems too good to be true, but at the very least there should be no harm trying to prove it. We have not yet found a counter example to this equality in a polynomial ring in characteristic zero. Another interesting question is under what conditions does the $\mathfrak{m}$-full closure of I have the form $I + \mathfrak{m}^k$ for some k. It is clear that this is not always true, but at least for complete intersections, there are many hints that suggest this should be true.

Finite Posets Versus Artinian Rings

As was mentioned in the preceding section there can be observed a strong parallelism between the theory of posets and that of commutative rings. The set S of monomials in the variables

$$x_1, x_2, \ldots, x_n$$

is a partially ordered set with divisibility as the order. It is a basis for the polynomial ring

$$R = K[x_1, x_2, \ldots, x_n]$$

as a linear space over a field K. The set of square-free monomials in S is the standard basis for the Artinian algebra

$$K[x_1, x_2, \ldots, x_n]/(x_1^2, x_2^2, \ldots, x_n^2).$$

An antichain in S is a minimal generating set of a monomial ideal of R. Let I be an ideal in R generated by monomials, and let P be the standard basis for R/I. Then P is a poset with rank function. This way of translation continues and a lot of ring-theoretic notions can find their counterpart in the theory of posets and conversely many combinatorial notions have ring-theoretic interpretations. Table 1 is a small dictionary of terms in the theory of Artinian rings and terms in the theory of finite sets, which interested readers may wish to expand.

One of the topics of this book is the Dilworth number of the Artinian rings of the form $A = R/I$, where the Dilworth number is defined to be the supremum of the number of generators of ideals in A. The definition of the Dilworth number for Artinian rings was introduced in [146] based on the theorem of Dilworth, as the definition for Artinian local rings is natural and is necessary to solve the problem of Rees. If I is generated by monomials, then the Dilworth number of R/I as an Artinian ring coincides with the Dilworth number of the poset $S \setminus I$ in the original sense. We want to know the number $\max\{\, \mu(J) \mid J \supset I \,\}$, which is the same as the Dilworth number of $A := R/\mathfrak{m}I$. The two Artinian rings A and $A' := R/I$ differ only by $I/\mathfrak{m}I$, which may be regarded as a set of generators for I and those

Table 1 A ring-poset dictionary of terms

Ring	Poset	Description
$R = K[x_1,\dots,x_n]$	$S = \{ x_1^{d_1} \cdots x_n^{d_n} \}$	S : a basis of R
Artinian ring	Finite poset	
Canonical module	Dual poset	
Hilbert function	Whitney numbers	
Degree	Rank	
Ideal	Filter	
R^*-ideal	ideal	$R^* = K[\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}]$
Minimal generating set	Antichain	
$\mathfrak{m}I$	Neighbor	$\mathfrak{m} = (x_1,\dots,x_n)$
Multiplication by l	Hasse diagram	$l = x_1 + \cdots + x_n$
Multiplication by $l + l^1 + l^2 + \cdots$	Reachability matrix	$l = x_1 + \cdots + x_n$
Standard basis for R/I	$S \setminus I$	I : an ideal of R
Dilworth number of R/I	Dilworth number of $S \setminus I$	I : an ideal of R
#1	Disjoint chain decomposition	
$R/(x_1^2,\dots,x_n^2)$	Boolean lattice	
$R/(x_1^{d_1},\dots,x_n^{d_n})$	Divisor lattice	
#2	Young diagram lattice	
#3	Vector space lattice	

See Sect. 3.5 (p. 126) for #1, Sect. 1.4.3 (p. 22) for #2, Sect. 1.4.4 (p. 29) for #3.

elements are the socle elements of A. So it is often the case that the Dilworth number of A is known from that of A'. (See Remark 2.44.)

Very fortunately the combinatorialists have developed a theory to determine the Dilworth number for various posets (e.g., [33, 34, 42]). In many important cases the Dilworth number turns out to be equal to the maximum of the Whitney numbers. (For Whitney numbers see Definition 1.24.) In such cases the poset is said to have the Sperner property. The following are basic examples of posets with rank function:

1. The Boolean lattice ($= 2^{[n]}$, all subsets of $[n] = \{ 1, 2, \dots, n \}$, see Example 1.2)
2. The divisor lattice or the finite chain product (see Sect. 1.4.2 for details)
3. The vector space lattice (see Sect. 1.4.4 for details)
4. The lattice of the Young diagrams contained in a rectangle (see Sect. 1.4.3)
5. The lattice of partitions of a set (see [42])

Among these five examples, (1)–(4) have the Sperner property. It was a long standing conjecture that (5) has the Sperner property [42], but in 1978 it was disproved by Canfield [14].

In 1928, Sperner [132] proved that the Boolean lattice has the Sperner property. More specifically, he proved that if F is a family of subsets of $\{ 1, 2, \dots, n \}$, in which no two members are related by inclusion, then $|F| \leq \binom{n}{[n/2]}$, and the equality occurs only in the case all sets have the same size.

This result was the origin of the Sperner theory. Once this was proved, it should have been easy to realize that the divisor lattice has the Sperner property. In spite of simple nature of this assertion, no proof was known for 20 years. In 1949, the Dutch Mathematical Society posed it as a prize problem, and in response three Dutch mathematicians gave a proof [11]. The idea of their proof is to decompose a divisor lattice into a disjoint union of "symmetric chains." (For details see, e.g., Aigner [2], Greene–Kleitman [42].)

Their method called "symmetric chain decomposition" resembles the Jordan block decomposition of a general element in the regular representation of an Artinian monomial complete intersection, and this seems to be a good evidence for the most natural method for proving the Sperner property for divisor lattice being the use of the theory of the special linear Lie algebra $\mathfrak{sl}_2$. This will be elucidated in Chap. 3.

The Hard Lefschetz Theorem and Sperner Property

Stanley [136] used the hard Lefschetz theorem to prove that certain partially ordered sets arising from a class of algebraic varieties have the Sperner property. To see an example, put

$$X = \mathbb{P}_{\mathbb{C}}^{d_1-1} \times \mathbb{P}_{\mathbb{C}}^{d_2-1} \times \cdots \times \mathbb{P}_{\mathbb{C}}^{d_n-1}.$$

The cohomology ring $H^*(X, \mathbb{C})$ of X is isomorphic to

$$\mathbb{C}[x_1, x_2, \ldots, x_n]/(x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n}).$$

The assertion of the hard Lefschetz theorem proves that the divisor lattice has the Sperner property. (In fact the hard Lefschetz theorem is a stronger statement than the Sperner property.) L. Reid, L. Roberts and M. Roitman [119] gave a purely algebraic proof for the equivalent statement to the hard Lefschetz theorem for the monomial complete intersection. One of the main objectives of this book is to provide a systematic method to prove the Sperner property for various posets including the four posets (1)–(4) above.

Schur–Weyl Duality and the Strong Lefschetz Property

The theory of the strong Lefschetz property for Artinian rings gives us a new way to look at representation theory of classical groups. A very basic fact of the representation of the general linear group $GL(d, K)$ is that all irreducible representations are obtained by decomposing the tensor representation

$$GL(d) \to GL(nd) = GL\left((K^d)^{\otimes n}\right)$$

into irreducible representations, and to decompose $(K^d)^{\otimes n}$ into irreducible $GL(d)$-modules is the "dual" to decomposing it into S_n-modules. To explain this in more detail, let the symmetric group S_n act on the tensor space $(K^d)^{\otimes n}$ by permutation of the components. A basic fact is that the actions of the two groups commute with each other. So the tensor space is in fact an $S_n \times GL(d)$-module, and an irreducible $S_n \times GL(d)$-module is of the form

$$F \otimes_K G,$$

where F is an irreducible S_n-module and G an irreducible $GL(d)$-module.

It is possible to identify the tensor space $(K^d)^{\otimes n}$ with the Artinian algebra:

$$A = K[x_1, x_2, \dots, x_n]/(x_1^d, x_2^d, \dots, x_n^d).$$

In fact, the isomorphism

$$(K^d)^{\otimes n} \to A = K[x_1, x_2, \dots, x_n]/(x_1^d, x_2^d, \dots, x_n^d)$$

is given by

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \mapsto x_1^{i_1 - 1} x_2^{i_2 - 1} \cdots x_n^{i_n - 1},$$

where $\{ e_1, \dots, e_d \}$ is the standard basis of K^d. The tensor space $(K^d)^{\otimes n}$ is acted on by the symmetric group S_n as permutation of components and an isotypic component of it as an S_n-module *is* an isotypic component of it as a $GL(d)$-module.

So, what is the advantage of considering the algebra A rather than the tensor space $(K^d)^{\otimes n}$? First, in the obvious sense A has a ring structure, so more information is available than just the tensor space $(K^d)^{\otimes n}$ offers. At least from the viewpoint of commutative algebraists, the algebra A should be easier to deal with than the tensor space.

Second, the action of the symmetric group S_n has a clear ring-theoretic meaning, namely, it is the permutation of the variables, where this action preserves the grading of A. One may realize that decomposing A into irreducible S_n-modules is more or less the same as decomposing the polynomial ring $K[x_1, x_2, \dots, x_n]$ into irreducible S_n-modules, which is well understood. For example, the ring of invariants R^{S_n} is the algebra generated by the elementary symmetric polynomials.

Third, the linear form $l := x_1 + x_2 + \cdots + x_n$ is an invariant of S_n and the multiplication map $\times l : A \to A$ as well as $\times l : R \to R$ preserves S_n-module structure. Thus, when the irreducible decomposition of R/lR is known, it is also known for R. The same is true for A/lA and A.

We have seen what the action of S_n means on A. On the other hand, one might ask what is the ring-theoretic meaning of the tensor representation,

$$\Phi : GL(d) \to GL\left((K^d)^{\otimes n}\right),$$

if we replace the tensor space by the Artinian ring $A = K[x_1, \ldots, x_n]/(x_1^d, \ldots, x_n^d)$. This can be explained as follows. Put $B = K[x]/(x^d)$ and let B^* be the multiplicative group of B. Similarly, let A^* be the multiplicative group of $A = K[x_1, \ldots, x_n]/(x_1^d, \ldots, x_n^d)$. Then we may define a group homomorphism

$$\Phi' : B^* \to A^*$$

by

$$f(x) \mapsto f(x_1)f(x_2)\cdots f(x_n).$$

With this notation we have the following commutative diagram of algebraic groups:

$$\begin{array}{ccc} GL(B) & \xrightarrow{\Phi} & GL(A) \\ \uparrow & & \uparrow \\ B^* & \xrightarrow{\Phi'} & A^* \end{array} \tag{4}$$

where the vertical maps are the group homomorphism induced by the regular representation of the algebras B and A. For example, the first vertical map is written as

$$f(x) \mapsto \begin{pmatrix} a_0 & & & & & \\ a_1 & a_0 & & & & \\ a_2 & a_1 & a_0 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ a_{d-2} & \vdots & \vdots & \ddots & \ddots & \\ a_{d-1} & a_{d-2} & a_{d-3} & \cdots & \cdots & a_0 \end{pmatrix},$$

where

$$f = a_0 + a_1x + a_2x^2 + \cdots + a_{d-1}x^{d-1} \in B.$$

We have explained the group homomorphism $\Phi' : B^* \to A^*$ and the vertical maps. So this explains ring-theoretically a part of

$$GL(d) \to GL(dn),$$

in the diagram (4) and it should give us a clear picture of the lower half of the tensor representation. As well as the multiplication by elements of A, the differential operator

$$a_0 + a_1 \frac{\partial}{\partial x} + \cdots + a_{d-1} \frac{\partial^{d-1}}{\partial x^{d-1}}$$

induces an automorphism of A (provided that $a_0 \neq 0$) and it explains the upper half of the tensor representation.

Historical Note by J. Watanabe

I learned the definition of "$\mathfrak{m}$-full ideal" as well as the associated problem (mentioned earlier) from Prof. D. Rees himself through a private conversation with him while he was visiting the Department of Mathematics of Nagoya University in 1983. He also showed me how one could define a "general element" for Artinian rings and showed me a proof for the statement in the Appendix of [147]. What it asserts is only too natural but the proof is very difficult. The proof written in Appendix of [147] and Theorem 5.1 and Proposition 5.2 of this book is due to Rees. Without the clear definition of general elements, the theory of Artinian rings could not start. Theorem 5.1 of this book is the basis of the theory of the SLP.

In [146], I used the term "strong Stanley property" for an Artinian Gorenstein graded algebra $A = \bigoplus_{i=0}^{c} A_i$ if there exists a linear form $l \in A$ such that the multiplication map $\times l^{c-2i} : A_i \to A_{c-i}$ is bijective for all $0 \leq i \leq [c/2]$, and other authors followed [66, 123]. It is an abstraction of the hard Lefschetz theorem and I came to this definition because it seemed to be the quickest way to prove that the maximum number $\mu(I)$ for ideals I in A is attained by a power of the maximal ideal. (It is a ring-theoretic interpretation of the Sperner property.) In 1990s the term "strong/weak Lefschetz property" began to be commonly used by many authors.

I would like to thank A. Iarrobino who showed his interest in the theory of Dilworth number of Artinian local rings at the earliest stage. Particularly, I would like to thank him for many discussions and encouragement. Special thanks are due to J. Migliore and U. Nagel for their invitation to let me share their idea to use the Grauert–Mülich theorem to prove that any complete intersection in embedding codimension three has the weak Lefschetz property [55]. It was the first breakthrough in an attempt to prove the Lefschetz property for complete intersections and it encouraged me greatly to continue my work on the strong and weak Lefschetz properties. H. Ikeda has contributed greatly to this theory. She constructed various examples of Gorenstein algebras without the SLP or WLP and gave a new proof for the SLP of the Boolean lattice [123], from which the Sperner property of monomial complete intersections follows. This is not well known but it preceded the proof of L. Reid, L. Roberts and M. Roitman [119].

Hiratsuka, Japan J. Watanabe

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