

# Real Algebraic Geometry

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Volume 66

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Vladimir I. Arnold

# Real Algebraic Geometry

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 Springer

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Originally published as “Veshchestvennaya algebraicheskaya geometriya,” MCCME (c) 2009

ISSN 2038-5722

ISSN 2038-5757 (electronic)

ISBN 978-3-642-36242-2

ISBN 978-3-642-36243-9 (eBook)

DOI 10.1007/978-3-642-36243-9

Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013933709

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*Cover design:* Beatrice B. Milano

Printed on acid-free paper

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# Publisher's Foreword

The preparations for this English-language edition of Vladimir Arnold's *Real Algebraic Geometry* began in the year 2009. With the sad and unexpected death of Arnold on June 3, 2010, publishing this book became a much more difficult task, and it was only with the tireless support and work of several of our collaborators and partners that publication became possible.

Vladimir Arnold had read a first part of the translation prepared by Gerald Gould, and in characteristic fashion, he had sent us, on November 1, 2009, a handwritten letter with his corrections and remarks. He never saw the first complete draft of the translation.

We then asked Ilia Itenberg, Viatcheslav Kharlamov, and Eugenio Shustin to act as editors for the book. We are very grateful to them and thank them for their excellent work. They not only read and checked all the mathematical details of the English edition, but supplied a set of end-notes and comments on the history of Gudkov's conjecture, which we hope will be appreciated by the readers of this book.

We thank Boris Khesin for drawing our attention to the original Russian edition of the book and for his help in enlisting the team of editors. Moreover, we thank Elionora Arnold for handling the formalities with the publishing agreement, thus clearing the legal path to publication.

Special thanks go to Ivan Yashchenko and Yuri Torkhov, of the Moscow Center for Continuous Mathematical Education. They readily agreed to Vladimir Arnold's request to include in this English edition an article of his that had originally been published in one of their journals; this now appears in Appendix A.

Reliably as always, David Kramer did the language editing of the text, and we thank him for his effort.

Finally, we thank our colleague Francesca Bonadei for arranging the publication of this book in the UNITEXT series.

*Martin Peters*



# Foreword

The book you have opened is not a systematic treatment of real algebraic geometry. It was designed as lecture notes destined for high-school students.

The various topics that are discussed in these notes have three features in common: they are about geometry; they treat objects that can be described by algebraic equations; and in most cases, the objects and equations are real. This certainly is one of the reasons for the title “Real Algebraic Geometry.” We conjecture that another—and perhaps even deeper—reason for the title is a certain polemic opposing *real* algebraic geometry, in the sense of the algebraic geometry that is closest to *reality*, to other kinds of algebraic geometry. Such a polemic is very much in Arnold’s spirit (cf. pp. 37–38).

The reader should not expect to find in this book a formal answer to questions such as, “What is real algebraic geometry?” and “What are its subject matter, main problems, and achievements?” To answer such questions is not the aim of this book (in fact, real algebraic geometry, as a mathematical field, is still in search of its identity).

Arnold has addressed the book to an open-minded reader who is ready to travel with him in a labyrinth of solved and open problems whose formulations are accessible to everyone with a basic knowledge of mathematics, but whose solutions require a certain ingenuity. The text is written in Arnold’s brilliant style. Explanations are transparent; the concepts and results are illustrated by numerous examples, interesting digressions to other areas of mathematics and to physics, historical facts, and anecdotes. At the same time, the text is not very polished. Perhaps Arnold wanted to preserve the lecture style of these notes (which certainly made the translator’s task difficult, especially since he attempted to replicate the original word for word as much as possible).

We found it necessary to provide comments at certain points in the text, and these appear as numbered endnotes. Finally, we are grateful to G. M. Polotovskiy for his help in collecting the materials related to the history of Gudkov’s conjecture (see “Editors’ Comments on Gudkov’s Conjecture,” which appear at the end of the book).

We are sure that the reader will enjoy this unordinary book.

*The editors*





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