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Poisson Structures

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For Clio

Preface

Poisson structures naturally appear in very different forms and contexts. Symplectic manifolds, Lie algebras, singularity theory, r-matrices, for example, all lead to a certain type of Poisson structures, sharing several features, despite the distances between the mathematics they originate from. This observation motivated us to bring the different worlds in which Poisson structures live together in a single volume, providing a multitude of entrances to the book, and hence to the subject. Thus, the idea of the body of the book, Part II, was born.

It is well known, i.e., it is commonly agreed upon by the experts, that results and techniques which are valid for one type of Poisson structure ought to apply, *mutatis mutandis*, to other types of Poisson structures. When starting to write Part I of the book, we soon realized that not everything could be derived from the general concept of a Poisson algebra, so we faced the challenge of presenting the concepts and the results in detail, for the algebraic, algebraic-geometric and geometric contexts, without copy-pasting large parts of the text two or three times. But as the writing moved on, both the algebraic and geometric contexts kept imposing themselves; finally, each found its proper place, appearing as being complementary and dual to the other, rather than a consequence or rephrasing, one of the other. It added, unexpectedly, an extra dimension to the book.

It was pointed out by one of the referees that the main applications of Poisson structures should be present in a book which has Poisson structures as its main subject. This was quite another challenge, giving rise to the third and final part of the book, Part III, undoubtedly an important addition.

Several years were necessary for this project, a big part was done at Poitiers, when the three of us were appointed to the "Laboratoire de Mathématiques et Applications". We were spending long hours together in what our colleagues called *The Aquarium* (for an explanation of the name, look up "poisson" in a French dictionary). When two of us moved away from Poitiers, we sometimes worked together in other places, which always offered us a pleasant and stimulating working atmosphere. Thus, we are happy to acknowledge the hospitality of our colleagues from the mathematics departments at the universities of Poitiers, Antwerp, Coimbra, Saint-Etienne and at the CRM in Barcelona and the Max-Planck Institute in Bonn. Each one of us was partially supported by an ANR contract (TcChAm, DPSing and GIMP, respectively), which made it possible to keep working on a blackboard, rather than seeing each other on a computer screen.

We learned Poisson structures from our teachers, friends and collaborators: Mark Adler, Paulo Antunes, Pantelis Damianou, Rui Fernandes, Benoit Fresse, Eva Miranda, Joana Nunes da Costa, Marco Pedroni, Michael Penkava, Claude Roger, Pierre van Moerbeke, Yvette Kosmann-Schwarzbach, Hervé Sabourin, Mathieu Stiénon, Friedrich Wagemann, Alan Weinstein, Ping Xu and Marco Zambon. The multitude of different points of view on the subject, which we learned from them, have given a quite specific flavor to the book. Even if a book cannot replace what one learns through lectures, discussions and collaborations, it is our hope that this book may further transfer what we learned from them and that it invites other researchers to explore the subject of Poisson structures, which turns out to be as diverse and rich as the colorful fauna and flora which inhabit the bottom of our oceans.

April 1, 2012 Camille Laurent-Gengoux, Anne Pichereau and Pol Vanhaecke

Contents

Introduction

Poisson structures appear in a large variety of different contexts, ranging from string theory, classical/quantum mechanics and differential geometry to abstract algebra, algebraic geometry and representation theory. In each one of these contexts, it turns out that the Poisson structure is not a theoretical artifact, but a key element which, unsolicited, comes along with the problem which is investigated and its delicate properties are in basically all cases decisive for the solution to the problem.

Hamiltonian mechanics and integrable systems. A first striking example of this phenomenon appears in classical mechanics. Poisson's classical bracket, which is given for smooth functions F and G on $\mathbb{R}^{2r}$ by

$$\{F,G\} := \sum_{i=1}^{r} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i} \frac{\partial F}{\partial p_i} \right) , \tag{0.1}$$

allows one to write Hamilton's equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, \ldots, r , \tag{0.2}$$

associated to a Hamiltonian $H = H(q, p)$, in a coordinate-free manner, treating positions q_i and momenta p_i on an equal footing. Indeed, using (0.1), the equations of motion (0.2) can be written as

$$\dot{q}_i = \{q_i, H\} , \qquad \dot{p}_i = \{p_i, H\} , \qquad i = 1, \ldots, r . \tag{0.3}$$

For every solution $t \mapsto (q(t), p(t))$ to these differential equations, it follows from (0.1) and (0.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t} F(q(t), p(t)) = \{F, H\} (q(t), p(t)) , \tag{0.4}$$

for all smooth functions F on $\mathbb{R}^{2r}$.

In differential geometrical terms, the Poisson bracket defines an operator which associates to every smooth function H on $\mathbb{R}^{2r}$ a vector field $\mathscr{X}_H$ on $\mathbb{R}^{2r}$. In terms of local coordinates, the vector field $\mathscr{X}_H$ is given by (0.3) and the variation of a smooth function F on $\mathbb{R}^{2r}$ along any integral curve $(q(t), p(t))$ of the vector field $\mathscr{X}_H$ is given by (0.4). In fact, when the function H is the Hamiltonian (the energy) of the system, the integral curves of $\mathscr{X}_H$ describe the time evolution of the system. Thus, the equations of motion of classical mechanics come along with an associated Poisson bracket; for simple systems, such as systems of particles with a classical interaction, the Poisson structure is the above one, but for constrained and reduced systems, the Poisson bracket takes a more complicated form.

The rôle which the Poisson structure plays in solving the problems of classical mechanics was known partly to Poisson, who pointed out that the vanishing of $\{F,H\}$ and $\{G,H\}$ implies the vanishing of $\{\{F,G\},H\}$. In mechanical terms, this means in view of (0.4) that the Poisson bracket of two constants of motion is again a constant of motion. Since constants of motion are a great help in the explicit integration of Hamilton's equations, Poisson's theorem brings to light the Poisson bracket as a tool for integration.

A strong amplification of this fact is the Liouville theorem, which states that r independent constants of motion $H_1 = H, H_2, \dots, H_r$ of (0.3) suffice for integrating Hamilton's equations by quadratures, under the hypothesis that these functions commute for the Poisson bracket (are in involution). A yet further amplification is the action-angle theorem, which states that in the neighborhood of a compact submanifold, traced out by the flows of the (commuting!) vector fields $\mathscr{X}_{H_i}$, there exist $(S^1)^r \times \mathbb{R}^r$-valued coordinates $(\theta_1, \dots, \theta_r, \rho_1, \dots, \rho_r)$ in which the Poisson bracket takes the canonical form (0.1), namely

$$\{F,G\} = \sum_{i=1}^{r} \left(\frac{\partial F}{\partial \theta_i} \frac{\partial G}{\partial \rho_i} - \frac{\partial G}{\partial \theta_i} \frac{\partial F}{\partial \rho_i} \right) ,$$

and such that the functions $H_1, \dots, H_r$ (including the Hamiltonian H) depend on the coordinates $\rho_1, \dots, \rho_r$ only. According to (0.2), Hamilton's equations take in these coordinates the simple form

$$\dot{\theta}_i = \frac{\partial H}{\partial \rho_i} , \qquad \dot{\rho}_i = -\frac{\partial H}{\partial \theta_i} = 0 , \qquad i = 1, \dots, r ,$$

which entails that the ρ_i are constant and hence that the θ_i are affine functions of time (since H does not depend on $\theta_1, \dots, \theta_r$).

Thus, in a nutshell, the equations of motion of classical mechanics come with a Poisson bracket, the Poisson bracket is decisive for their integrability and it permits us to construct coordinates in which the problem (including the Poisson bracket) takes a very simple form, from which the solutions and their characteristics can be read off at once.

Abstraction and generalization. Before giving another example of the key rôle played by Poisson brackets, we explain the algebraic and geometric abstraction of the classical Poisson bracket (0.1). It is well known that Poisson's theorem is explained by the Jacobi identity

$$\{\{F,G\},H\}+\{\{G,H\},F\}+\{\{H,F\},G\}=0\,, \tag{0.5}$$

valid for arbitrary smooth functions F,G and H on $\mathbb{R}^{2r}$. It is however noteworthy to point out that Jacobi discovered this identity only thirty years after Poisson announced his theorem. Taking arbitrary smooth functions $x_{ij}=-x_{ji}$ defined on a non-empty open subset of $\mathbb{R}^d$, with coordinates $x_1,\dots,x_d$, the skew-symmetric bilinear operation $\{\cdot,\cdot\}$, defined on smooth functions by

$$\{F,G\}:=\sum_{i,j=1}^{d} x_{ij}\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j}\,, \tag{0.6}$$

satisfies the Jacobi identity if and only if the functions x_{ij} satisfy the identity

$$\sum_{\ell=1}^{d}\left(x_{\ell k}\frac{\partial x_{ij}}{\partial x_\ell}+x_{\ell i}\frac{\partial x_{jk}}{\partial x_\ell}+x_{\ell j}\frac{\partial x_{ki}}{\partial x_\ell}\right)=0\,.$$

This fact was observed by Lie, who also pointed out that under the assumption of constancy of the rank (which he assumes implicitly), there exist local coordinates $q_1,\dots,q_r,p_1,\dots,p_r,z_1,\dots,z_s$, where $d=2r+s$, such that (0.6) takes the form

$$\{F,G\}=\sum_{i=1}^{r}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}-\frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}\right)\,,$$

which is formally the same as (0.1), but on a $d=2r+s$ dimensional space, with s of the variables being totally absent from the bracket. An important class of examples of brackets (0.6), satisfying (0.5), is defined on the dual $\mathfrak{g}^*$ of a (finite-dimensional) Lie algebra $\mathfrak{g}$, where the bracket is directly inherited by the Lie bracket. Many important facts about the Poisson bracket on $\mathfrak{g}^*$, such as the relation to the coadjoint representation of the adjoint group $\mathbf{G}$ of $\mathfrak{g}$, were observed by Lie and were rediscovered and further expanded by Berezin [21], Kirillov [105, 106], Kostant [117] and Souriau [185].

It was pointed out by Lichnerowicz in [126] that, in abstract differential geometrical terms, a Poisson structure on a smooth manifold M is a smooth bivector field π on M, satisfying $[\pi,\pi]_S=0$, where $[\cdot,\cdot]_S$ stands for the Schouten bracket (a natural extension of the Lie bracket of vector fields to arbitrary multivector fields). It implies that for every open subset U of M, the bilinear operation $\{\cdot,\cdot\}_U$, defined for $F,G\in C^\infty(U)$ by

$$\{F,G\}_U(m):=\langle \mathrm{d}_mF\wedge \mathrm{d}_mG,\pi_m\rangle\,, \tag{0.7}$$

for all $m\in M$, satisfies the Jacobi identity, hence defines a Lie algebra structure on $C^\infty(U)$.

Moreover, this Lie algebra structure on $C^\infty(U)$ and the associative structure on $C^\infty(U)$ (pointwise multiplication of functions) are compatible in the sense that

$$\{FG,H\}_U = F\{G,H\}_U + G\{F,H\}_U \ ,$$

for all $F,G,H \in C^\infty(U)$.

It is from here easy to make the algebraic abstraction of the notion of a Poisson structure, leading to the notion of a Poisson algebra. For concreteness, we give the definition in the case of a vector space $\mathscr{A}$ over a field $\mathbb{F}$ of characteristic zero (one may think of $\mathbb{R}$ or $\mathbb{C}$). Two products (bilinear maps from $\mathscr{A}$ to itself) are assumed to be given on $\mathscr{A}$, one denoted by "$\cdot$", which is assumed to be associative and commutative, while the other one, denoted by $\{\cdot\,,\cdot\}$, is assumed to be a Lie bracket. The triple $(\mathscr{A},\cdot,\{\cdot\,,\cdot\})$ is called a Poisson algebra if the two products are compatible in the sense that

$$\{F\cdot G,H\} = F\cdot\{G,H\} + G\cdot\{F,H\} \ , \tag{0.8}$$

for all $F,G,H \in \mathscr{A}$. It is clear that in the case of a Poisson manifold (M,π), for each open subset U of M, the triple $(C^\infty(U),\cdot,\{\cdot\,,\cdot\}_U)$ is a Poisson algebra, where $\cdot$ stands for the pointwise product of functions and $\{\cdot\,,\cdot\}_U$ stands for the Lie bracket defined by (0.7).

Deformation theory and quantization. We now come to a second striking example of Poisson structures making an unexpected appearance, and playing next a very definite rôle in the statement of the problem and in its final solution. We give a purely algebraic, but very simple and natural, introduction to the problem. In this approach, one speaks of "deformation theory". A physical interpretation of the problem will be given afterwards; one then speaks of "quantization" or of "deformation quantization".

We start with a commutative associative algebra $\mathscr{A} = (\mathscr{A},\cdot)$. The reader may think of the example of the polynomial algebra $\mathscr{A} = \mathbb{F}[x_1,\dots,x_d]$. We suppose that the product in $\mathscr{A}$ is part of a family of associative (not necessarily commutative) products, say we have a family $\star_t$ of bilinear maps

$$\begin{array}{rcl} \star_t : \mathscr{A}\times\mathscr{A} & \to & \mathscr{A} \\ (a,b) & \mapsto & a\star_t b \end{array} \tag{0.9}$$

such that, for each fixed $t \in \mathbb{F}$, the product $\star_t$ is associative, with $\star_0$ being the original associative commutative product on $\mathscr{A}$, i.e., $(\mathscr{A},\cdot) = (\mathscr{A},\star_0)$. We refer to the family of associative products $\star_t$ on $\mathscr{A}$ as a *deformation* of $\mathscr{A}$. It is assumed that the family depends *nicely* on t, in a way which we do not make precise here. We consider, for every $t \in \mathbb{F}$, the commutator of $\star_t$, which is defined for all $a,b \in \mathscr{A}$ by

$$[a,b]_t := a\star_t b - b\star_t a \ .$$

Since $\star_t$ is associative, the commutator $[\cdot\,,\cdot]_t$ satisfies for each $t \in \mathbb{F}$ the following identities, valid for all $a,b,c \in \mathscr{A}$:

$$
\begin{aligned}
\left[a \star_t b, c\right]_t &= a \star_t \left[b,c\right]_t + \left[a,c\right]_t \star_t b \,, \\
\left[\left[a,b\right]_t, c\right]_t + \left[\left[b,c\right]_t, a\right]_t + \left[\left[c,a\right]_t, b\right]_t &= 0 \,.
\end{aligned}
\tag{0.10}
$$

Consider the skew-symmetric bilinear map $\{\cdot\,,\cdot\} : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$, which is defined for $a,b \in \mathscr{A}$ by

$$
\{a,b\} := \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \left[a,b\right]_t = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \left(a \star_t b - b \star_t a\right) \,.
$$

Then the first equation in (0.10) implies that $\{\cdot\,,\cdot\}$ satisfies the Leibniz property (0.8), while the second equation in (0.10) implies that $\{\cdot\,,\cdot\}$ satisfies the Jacobi identity. Combined, it means that $(\mathscr{A},\cdot,\{\cdot\,,\cdot\})$ is a Poisson algebra. Thus, a family of deformations of a given commutative associative algebra leads naturally to a Poisson bracket on this algebra! In the particular case of the associative algebras of operators of quantum mechanics, in which t is a multiple of Planck's constant $\hbar$, the operators become in the limit $\hbar \to 0$ commuting classical variables and the above procedure yields a Poisson bracket on the algebra of classical variables.

The main problem of deformation theory is to invert this procedure; in the context of physics, this is called the quantization problem or, more precisely, the problem of quantization by deformation. In its algebraic version, the first question is whether given a Poisson algebra $(\mathscr{A},\cdot,\{\cdot\,,\cdot\})$ there exists a family of associative products $\star_t$ on $\mathscr{A}$, such that $(\mathscr{A},\cdot) = (\mathscr{A},\star_0)$ and such that $\{\cdot\,,\cdot\}$, obtained from $\star_t$ by the above procedure, is a Poisson bracket on $\mathscr{A}$. The second question is to classify all such deformations, for a given $(\mathscr{A},\cdot,\{\cdot\,,\cdot\})$. The physical interpretation of these questions is that, starting from the algebra of classical variables, which inherits from phase space a Poisson bracket, as explained above in the context of mechanical systems, one wishes to (re-)construct the corresponding algebra of quantum mechanical operators.

A mathematical simplification of this problem is to ask whether the above procedure can be inverted formally. Stated in a precise way, this means the following. Let $(\mathscr{A},\cdot,\{\cdot\,,\cdot\})$ be a Poisson algebra and consider the vector space $\mathscr{A}[[\nu]]$ of formal power series in one variable ν, whose coefficients belong to $\mathscr{A}$. Let $\mu_\star$ denote an associative product on $\mathscr{A}[[\nu]]$, i.e., an $\mathbb{F}[[\nu]]$-bilinear map from $\mathscr{A}[[\nu]]$ to itself, which is associative. Given $a,b \in \mathscr{A}$, we can write

$$
\mu_\star(a,b) = \sum_{k \in \mathbb{N}} \mu_k(a,b)\, \nu^k \,,
$$

which defines bilinear maps $\mu_i : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ for $i \in \mathbb{N}$. By definition, $\mu_\star$ is a formal deformation of $(\mathscr{A},\cdot,\{\cdot\,,\cdot\})$ if $\mu_0(a,b) = a \cdot b$ and $\mu_1(a,b) - \mu_1(b,a) = \{a,b\}$, for all $a,b \in \mathscr{A}$. The main question, stated formally, is then if every Poisson algebra $(\mathscr{A},\cdot,\{\cdot\,,\cdot\})$ admits a formal deformation $\mu_\star$.

Consider for example $\mathbb{R}^{2r}$, with coordinates $(x_1,\dots,x_{2r}) = (q_1,\dots,q_r,p_1,\dots,p_r)$. The *Moyal product* is the $\mathbb{R}[[\nu]]$-bilinear product on $C^\infty(\mathbb{R}^{2r})[[\nu]]$, which is given for $F,G \in C^\infty(\mathbb{R}^{2r})$ by

$$\mu_\star(F,G) := \sum_{k\in\mathbb{N}} \sum_{1\leqslant i_1,j_1,\dots,i_k,j_k\leqslant 2r} J_{i_1,j_1}\dots J_{i_k,j_k} \frac{\partial^k F}{\partial x_{i_1}\dots\partial x_{i_r}} \frac{\partial^k G}{\partial x_{j_1}\dots\partial x_{j_r}} \frac{\nu^k}{k!} ,$$

where J is the skew-symmetric matrix of size $2r$, given by $J := \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix}$. The Moyal product, also called the Moyal–Weyl product, defines an associative product on $C^\infty(\mathbb{R}^{2r})[[\nu]]$, whose leading terms satisfy $\mu_0(F,G) = FG$ and

$$\mu_1(F,G) - \mu_1(G,F) = \sum_{j=1}^{r} \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial G}{\partial q_j}\frac{\partial F}{\partial p_j} \right) ,$$

where the latter right-hand side is the Poisson bracket (0.1). This shows that the classical Poisson bracket admits a formal deformation.

It was shown by De Wilde and Lecomte in [57] that the Poisson algebra of an arbitrary symplectic manifold (regular Poisson manifolds of maximal rank) admits a formal deformation. A geometrical proof of their result, which also works in the case of arbitrary regular Poisson manifolds, was given by Fedosov [73]. Finally, it was Kontsevich [107] who proved, using ideas which come from string theory, that the algebra of functions on any Poisson manifold admits a formal deformation. He proved in fact a quite stronger result, which says that for a given Poisson manifold (M,π), the equivalence classes of formal deformations of the algebra of smooth functions on M are classified by the equivalence classes of formal deformations of the Poisson structure π.

Summarizing, when one deforms a commutative associative algebra, a Poisson structure shows up and it plays a dominant rôle in the entire deformation process.

The examples. The main idea which led us to the writing of this book, and to its actual structure, is that Poisson structures come in big classes (families) where roughly speaking each class has its own tools and is related to a very definite part of mathematics, hence leading to specific questions. As a consequence, the main part of the book (about half of it) is Part II, dedicated to six classes of examples. They make their appearance in six different chapters, namely Chapters 6–11, as follows.

• Constant Poisson structures, such as Poisson's original bracket (0.1), are according to the Darboux theorem the model (normal form) for Poisson structures on real or complex manifolds, in the neighborhood of any point where the rank is locally constant. Regular Poisson manifolds and symplectic manifolds (such as cotangent bundles and Kähler manifolds) are the main examples, which exhibit even in the absence of singularities some of the main phenomena in Poisson geometry, which distinguish it from Riemannian geometry and show its pertinence for classical and quantum mechanics. Constant and regular Poisson structures are studied in Chapter 6.

• Linear Poisson structures are in one-to-one correspondence with Lie algebras and many features of Lie algebras are most naturally approached in terms of the canonical Poisson bracket on the dual of a (finite-dimensional) Lie algebra, the so-called Lie–Poisson structure. The coadjoint orbits, for example, of a Lie algebra are the symplectic leaves of the Lie–Poisson structure, showing on the one hand that they are even dimensional, and on the other hand that they carry a canonical symplectic structure, the Kostant–Kirillov–Souriau symplectic structure. Linear Poisson structures appear also on an arbitrary Poisson manifold (M,π) at any point x where the Poisson structure vanishes: the tangent space T_xM inherits a linear Poisson structure, the linearization of π at x. It leads to the linearization problem, which inquires if π and its linearization at x are isomorphic, at least on a neighborhood of x. Linear Poisson structures and their relation to Lie algebras are the subject of Chapter 7.

• Many Poisson structures of interest are neither constant nor linear (or affine). In fact, while the basic properties of the latter Poisson structures can be traced back to (known) properties of bilinear forms and of Lie algebras, new phenomena appear when considering quadratic Poisson structures (i.e., homogeneous Poisson structures of degree two) and higher degree Poisson structures, which often turn out to be weight homogeneous. A prime example of this is the transverse Poisson structure to an adjoint orbit in a semi-simple Lie algebra. Moreover, this Poisson structure arises in the case of the subregular orbit from a Nambu–Poisson structure, defined by the invariant functions of the Lie algebra. Higher degree Poisson structures are studied in Chapter 8.

• In dimension two, every bivector field is a Poisson structure, yet many questions about Poisson structures on surfaces are non-trivial. For example, their local classification has up to now only been accomplished under quite strong regularity assumptions on the singular locus of the Poisson structure. In dimension three, the Jacobi identity can be stated as an integrability condition of a distribution (or of a one-form), which eventually leads to the symplectic foliation. A surface in $\mathbb{C}^3$ which is defined by the zero locus of a polynomial φ inherits a Poisson structure from the standard Nambu–Poisson structure on $\mathbb{C}^3$, with φ as Casimir. In the case of singular surfaces $\mathbb{C}^2/\mathbf{G}$, where $\mathbf{G}$ is a finite subgroup of $\mathbf{SL}_2(\mathbb{C})$, this Poisson structure coincides with the Poisson structure obtained from the canonical symplectic structure on $\mathbb{C}^2$ by reduction. Thus, rather than being trivial, Poisson structures in dimensions two and three form a rich playground on which a variety of phenomena can be observed. Poisson structures in dimensions two and three are discussed in Chapter 9.

• In a Lie-theoretical context, a linear Poisson structure, different from the canonical Lie–Poisson structure, often pops up. This Poisson structure appears on the Lie algebra $\mathfrak{g}$ at hand, so that the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ comes equipped with a Lie algebra structure, or it appears on $\mathfrak{g}^*$, so that $\mathfrak{g}$ is equipped with a second Lie algebra structure. The underlying operator, which relates either of these Lie structures with the original Lie bracket on $\mathfrak{g}$ is in the first case an element $r \in \mathfrak{g} \otimes \mathfrak{g}$, called an r-matrix, while it is in the second case a (vector space) endomorphism R of $\mathfrak{g}$, called an R-matrix. These structures appeared first in the theory of integrable systems, but are nowadays equally important in the theory of Lie–Poisson groups (see the next item) and of quantum groups. In the context of Lie algebras which come from associative algebras, r-matrices and R-matrices also lead to a quadratic Poisson structure on the Lie algebra, which plays an important rôle in the theory of Lie–Poisson groups; they also lead to a cubic Poisson structure, whose virtue seems at this point still very mysterious. Poisson structures coming from r-matrices or R-matrices are the subject of Chapter 10.

• A Poisson–Lie group is a Lie group $\mathbf{G}$, equipped with a Poisson structure π for which the group multiplication $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$ is a Poisson map. The Poisson structure π leads to a Lie algebra structure on the dual of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$, making $\mathfrak{g}$ into a Lie bialgebra, a structure generalizing the structure which comes from an r-matrix. Conversely, every finite-dimensional Lie bialgebra is obtained in this way from a Poisson–Lie group, a result which extends Lie's third theorem (every finite-dimensional Lie algebra is the Lie algebra of a Lie group). Poisson–Lie groups have many applications, for example in the description of the Schubert cells of a Lie group. Poisson–Lie groups are discussed in Chapter 11.

Theoretical foundations. The examples (Chapters 6–11) are preceded by a first part (5 chapters) with a systematic exposition of the general theory of Poisson structures. We used two principles in the writing of these chapters. The first one is that we wanted to develop both the algebraic and geometric points of view of the theory. In fact, there has for the last decade been an increasing interest in the algebraic aspects of Poisson structures, while of course geometrical intuition underlies all constructions and the vast majority of examples. In the algebraic context, we have a Poisson algebra, whose underlying space is an $\mathbb{F}$-vector space, where $\mathbb{F}$ is an arbitrary field of characteristic zero. In the geometric context, we have a Poisson manifold, where the manifold is either real or complex. In between the algebraic and geometric contexts, Poisson varieties appear, whose underlying object can both be viewed as a variety or as a finitely generated algebra. Throughout this first part, the reader will experience that although most of the results are formally very similar in the algebraic and in the geometric setting, their concrete implementation (including proofs of the propositions) is quite different and needs to be worked out in detail in both settings.

The second principle which we used was to keep the prerequisites in (commutative and Lie) algebra and in (differential and algebraic) geometry to a minimum. Thus, for example, when we describe Poisson structures geometrically as bivector fields, we explain the notion of a bivector field by first recalling the notion of a vector field, rather than simply saying that a bivector field is a section of the exterior square of the tangent bundle to the manifold. Similarly, while the basic facts about Lie groups and Lie algebras are supposed to be known, the interplay between Lie groups and Lie algebras is recalled in detail, as it is further amplified in the presence of Poisson structures. We added an appendix at the end of the book with some facts on multilinear algebra (tensor products, wedges, algebra and coalgebra structures) and an appendix which recalls the basic facts about differential geometry. The objects and properties which are recalled in these appendices are used throughout the book.

Chapters 1 and 2 contain the basic definitions and constructions. A good familiarity with these chapters should suffice for reading a good part of any other chapter of the book. Chapter 3 deals with multi-derivations and their geometrical analog, multivector fields, and with Kähler forms, which are the algebraic analogs of differential forms. Chapter 4 deals with Poisson cohomology and Chapter 5 is devoted to reduction in the context of Poisson structures. At the end of each of these five chapters, we give a list of exercises, so that the reader can test his understanding of the theory.

Applications. The third part of the book contains the two major applications, which we discussed above: integrable systems are discussed in Chapter 12, while deformation quantization is detailed in Chapter 13. They are however not the only applications which are given in the book. In fact, we end each example chapter (Chapters 6–11) with an application of the class of Poisson structures, studied in that chapter.

What is absent from this book. The main topic about Poisson structures which is absent from this book is what should be called "Poisson geometry". By this we mean the global geometry of Poisson structures, which involves the integration of Poisson brackets, Poisson connections, stability of symplectic leaves and related topics. For this, we refer to [51, 53, 74].

Equally absent from this book are the many generalizations of Poisson structures. Quasi-Poisson structures and Poisson structures with background are not only mathematically speaking very interesting, see [112], they have in addition non-trivial applications in physics, see [119, 162, 176], and are themselves a particular case of Dirac structures [48, 89]. The theory of quantum groups, initiated in the seminal ICM talk [59] by Drinfel'd, has several connections with the topics which are developed in this book, but adding several chapters to the book would not have been sufficient to give a fair account of this subject; we simply refer to the excellent books [40, 103].

Equally absent from this book is the theory of Lie algebroids, introduced by Pradines [171], which are both a particular case and a generalization of Poisson manifolds. In particular, after the pioneering work of [47], the problem of the integration of Poisson manifolds, which claims that symplectic Lie groupoids are to Poisson manifolds what Lie groups are to Lie algebras, has gained a long attention. For a definite solution of this problem, given by Cranic and Fernandes, see [49, 50].

The notes and the references. When writing the history of the theory of Poisson brackets, giving each of the main results in the theory and examples the "right" credit is a challenge which is beyond the scope of this book. We claim no originality in this book, but since none of the proofs which are given are copy-and-paste proofs from existing proofs, we did not think it was necessary to cite the proof in the literature which is closest to the proof that we give. Also, many properties of Poisson structures, and many examples, have been through a long series of transformations and generalizations, think for example of Weinstein's splitting theorem or the Poisson and Poisson–Dirac reduction theorems; structuring the long list of important intermediate results which led to the final results is certainly interesting from the epistemological point of view, but this was not our main focus when writing the book.

However, at the end of each chapter, we have put a section called "Notes", in which we indicate some references, including the main references which we know of, for further reading on what has been treated in the chapter. These notes are also used to situate, a posteriori, the treated subjects in the scientific literature and to trace some of their historical developments. Finally, the notes are also used to indicate some further connections to other fields of mathematics or physics. Since Poisson structures are a very active field of research, it is clear that the list of connections with other fields is not exhaustive.