Differential-Algebraic Equations Forum

Differential-Algebraic Equations Forum

The series "Differential-Algebraic Equations Forum" is concerned with analytical, algebraic, control theoretic and numerical aspects of differential algebraic equations (DAEs) as well as their applications in science and engineering. It is aimed to contain survey and mathematically rigorous articles, research monographs and textbooks. Proposals are assigned to an Associate Editor, who recommends publication on the basis of a detailed and careful evaluation by at least two referees. The appraisals will be based on the substance and quality of the exposition.

For further volumes:
www.springer.com/series/11221

René Lamour • Roswitha März • Caren Tischendorf

Differential-Algebraic Equations: A Projector Based Analysis

Springer

René Lamour
Department of Mathematics
Humboldt-University of Berlin
Berlin, Germany

Caren Tischendorf
Mathematical Institute
University of Cologne
Cologne, Germany

Roswitha März
Department of Mathematics
Humboldt-University of Berlin
Berlin, Germany

ISBN 978-3-642-27554-8 ISBN 978-3-642-27555-5 (eBook)
DOI 10.1007/978-3-642-27555-5
Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013930590

Mathematics Subject Classification (2010): 34A09, 34-02, 65L80, 65-02, 15A22, 65L05, 49K15, 34D05, 34G99

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Foreword by the Editors

We are very pleased to write the Foreword of this book by René Lamour, Roswitha März, and Caren Tischendorf. This book appears as the first volume in the recently established series "FORUM DAEs"—a forum which aims to present different directions in the widely expanding field of differential-algebraic equations (DAEs).

Although the theory of DAEs can be traced back earlier, it was not until the 1960s that mathematicians and engineers started to study seriously various aspects of DAEs, such as computational issues, mathematical theory, and applications. DAEs have developed today, half a century later, into a discipline of their own within applied mathematics, with many relationships to mathematical disciplines such as algebra, functional analysis, numerical analysis, stochastics, and control theory, to mention but a few. There is an intrinsic mathematical interest in this field, but this development is also supported by extensive applications of DAEs in chemical, electrical and mechanical engineering, as well as in economics.

Roswitha März' group has been at the forefront of the development of the mathematical theory of DAEs since the early 1980s; her valuable contribution was to introduce—with a Russian functional analytic background—the method now known as the "projector approach" in DAEs. Over more than 30 years, Roswitha März established a well-known group within the DAE community, making many fundamental contributions. The projector approach has proven to be valuable for a huge class of problems related to DAEs, including the (numerical) analysis of models for dynamics of electrical circuits, mechanical multibody systems, optimal control problems, and infinite-dimensional differential-algebraic systems.

Broadly speaking, the results of the group have been collected in the present textbook, which comprises 30 years of development in DAEs from the viewpoint of projectors. It contains a rigorous and stand-alone introduction to the projector approach to DAEs. Beginning with the case of linear constant coefficient DAEs, this approach is then developed stepwise for more general types, such as linear DAEs with variable coefficients and nonlinear problems. A central concept in the theory of DAEs is the "index", which is, roughly speaking, a measure of the difficulty of (numerical) solution of a given DAE. Various index concepts exist in the theory of DAEs; and the one related to the projector approach is the "tractability index". Analytical and numerical consequences of the tractability index are presented. In addition to the discussion of the analytical and numerical aspects of different classes of DAEs, this book places special emphasis on DAEs which are explicitly motivated by practice: The "functionality" of the tractability index is demonstrated by means of DAEs arising in models for the dynamics of electrical circuits, where the index has an explicit interpretation in terms of the topological structure of the interconnections of the circuit elements. Further applications and extensions of the projector approach to optimization problems with DAE constraints and even coupled systems of DAEs and partial differential equations (the so-called "PDAEs") are presented.

If one distinguishes strictly between a textbook and a monograph, then we consider the present book to be the second available textbook on DAEs. Not only is it complementary to the other textbook in the mathematical treatment of DAEs, this book is more research-oriented than a tutorial introduction; novel and unpublished research results are presented. Nonetheless it contains a self-contained introduction to the projector approach. Also various relations and substantial cross-references to other approaches to DAEs are highlighted.

This book is a textbook on DAEs which gives a rigorous and detailed mathematical treatment of the subject; it also contains aspects of computations and applications. It is addressed to mathematicians and engineers working in this field, and it is accessible to students of mathematics after two years of study, and also certainly to lecturers and researchers. The mathematical treatment is complemented by many examples, illustrations and explanatory comments.

Ilmenau, Germany *Achim Ilchmann*
Hamburg, Germany *Timo Reis*
June 2012

Preface

We assume that differential-algebraic equations (DAEs) and their more abstract versions in infinite-dimensional spaces comprise *great potential for future mathematical modeling*. To an increasingly large extent, in applications, DAEs are automatically generated, often by coupling various subsystems with large dimensions, but *without manifested mathematically useful structures*. Providing tools to uncover and to monitor mathematical DAE structures is one of the current challenges. What is needed are criteria in terms of the original data of the given DAE. The projector based DAE analysis presented in this monograph is intended to address these questions.

We have been working on our theory of DAEs for quite some time. This theory has now achieved a certain maturity. Accordingly, it is time to record these developments in one coherent account. From the very beginning we were in the fortunate position to communicate with colleagues from all over the world, advancing different views on the topic, starting with Linda R. Petzold, Stephen L. Campbell, Werner C. Rheinboldt, Yuri E. Boyarintsev, Ernst Hairer, John C. Butcher and many others not mentioned here up to John D. Pryce, Ned Nedialkov, Andreas Griewank. We thank all of them for stimulating discussions.

For years, all of us have taught courses, held seminars, supervised diploma students and PhD students, and gained fruitful feedback, which has promoted the progress of our theory. We are indebted to all involved students and colleagues, most notably the PhD students.

Our work was inspired by several fascinating projects and long term cooperation, in particular with Roland England, Uwe Feldmann, Claus Führer, Michael Günther, Francesca Mazzia, Volker Mehrmann, Peter C. Müller, Peter Rentrop, Ewa Weinmüller, Renate Winkler.

We very much appreciate the joint work with Katalin Balla, who passed away too early in 2005, and the colleagues Michael Hanke, Immaculada Higueras, Galina Kurina, and Ricardo Riaza. All of them contributed essential ideas to the projector based DAE analysis.

We are indebted to the German Federal Ministry of Education and Research (BMBF) and the German Research Foundation (DFG), in particular the research center MATHEON in Berlin, for supporting our research in a lot of projects.

We would like to express our gratitude to many people for their support in the preparation of this volume. In particular we thank our colleague Jutta Kerger.

Last but not least, our special thanks are due to Achim Ilchmann and Timo Reis, the editors of the DAE Forum. We appreciate very much their competent counsel for improving the presentation of the theory.
We are under obligations to the staff of Springer for their careful assistance.

René Lamour Roswitha März Caren Tischendorf

Contents

Notations . . . xv

Introduction . . . xix

Part I Projector based approach

1 Linear constant coefficient DAEs . . . 3
1.1 Regular DAEs and the Weierstraß–Kronecker form . . . 3
1.2 Projector based decoupling of regular DAEs . . . 10
1.2.1 Admissible matrix sequences and admissible projectors . . . 10
1.2.2 Decoupling by admissible projectors . . . 23
1.2.3 Complete decoupling . . . 30
1.2.4 Hierarchy of projector sequences for constant matrix pencils . . . 36
1.2.5 Compression to a generalized Weierstraß–Kronecker form . . . 37
1.2.6 Admissible projectors for matrix pairs in a generalized Weierstraß–Kronecker form . . . 41
1.3 Transformation invariance . . . 45
1.4 Characterizing matrix pencils by admissible projectors . . . 47
1.5 Properly stated leading term and solution space . . . 50
1.6 Notes and references . . . 52

2 Linear DAEs with variable coefficients . . . 57
2.1 Properly stated leading terms . . . 58
2.2 Admissible matrix function sequences . . . 60
2.2.1 Basics . . . 60
2.2.2 Admissible projector functions and characteristic values . . . 65
2.2.3 Widely orthogonal projector functions . . . 75
2.3 Invariants under transformations and refactorizations . . . 79
2.4 Decoupling regular DAEs . . . 86
2.4.1 Preliminary decoupling rearrangements . . . 86

2.4.2 Regularity and basic decoupling of regular DAEs 90
2.4.3 Fine and complete decouplings 104
2.4.3.1 Index-1 case 104
2.4.3.2 Index-2 case 106
2.4.3.3 General benefits from fine decouplings 108
2.4.3.4 Existence of fine and complete decouplings 111
2.5 Hierarchy of admissible projector function sequences for linear DAEs 117
2.6 Fine regular DAEs 118
2.6.1 Fundamental solution matrices 119
2.6.2 Consistent initial values and flow structure 123
2.6.3 Stability issues 127
2.6.4 Characterizing admissible excitations and perturbation index 132
2.7 Specifications for regular standard form DAEs 137
2.8 The T-canonical form 140
2.9 Regularity intervals and critical points 146
2.10 Strangeness versus tractability 160
2.10.1 Canonical forms 160
2.10.2 Strangeness reduction 164
2.10.3 Projector based reduction 166
2.11 Generalized solutions 171
2.11.1 Measurable solutions 171
2.11.2 Distributional solutions 173
2.12 Notes and references 174

3 Nonlinear DAEs 183
3.1 Basic assumptions and notions 184
3.1.1 Properly involved derivative 184
3.1.2 Constraints and consistent initial values 187
3.1.3 Linearization 195
3.2 Admissible matrix function sequences and admissible projector functions 198
3.3 Regularity regions 208
3.4 Transformation invariance 224
3.5 Hessenberg form DAEs of arbitrary size 229
3.6 DAEs in circuit simulation 239
3.7 Local solvability 250
3.7.1 Index-1 DAEs 251
3.7.2 Index-2 DAEs 259
3.7.2.1 Advanced decoupling of linear index-2 DAEs 259
3.7.2.2 Nonlinear index-2 DAEs 261
3.7.2.3 Index reduction step 268
3.8 Advanced localization of regularity: including jet variables 272
3.9 Operator settings 281

3.9.1 Linear case ... 283
3.9.2 Nonlinear case ... 287
3.10 A glance at the standard approach via the derivative array and differentiation index ... 290
3.11 Using structural peculiarities to ease models ... 300
3.12 Regularity regions of DAEs with quasi-proper leading terms ... 304
3.13 Notes and references ... 307

Part II Index-1 DAEs: Analysis and numerical treatment

4 Analysis ... 317
4.1 Basic assumptions and notions ... 317
4.2 Structure and solvability of index-1 DAEs ... 320
4.3 Consistent initial values ... 334
4.4 Notes and references ... 336

5 Numerical integration ... 339
5.1 Basic idea ... 340
5.2 Methods applied to ODEs and DAEs in standard form ... 345
5.2.1 Backward differentiation formula ... 345
5.2.2 Runge–Kutta method ... 346
5.2.3 General linear method ... 350
5.3 Methods applied to DAEs with a properly involved derivative ... 352
5.3.1 Backward differentiation formula ... 352
5.3.2 Runge–Kutta method ... 353
5.3.3 General linear method ... 355
5.4 When do decoupling and discretization commute? ... 356
5.5 Convergence on compact intervals and error estimations ... 361
5.5.1 Backward differentiation formula ... 361
5.5.2 IRK(DAE) method ... 364
5.5.3 General linear method ... 369
5.6 Notes and references ... 371

6 Stability issues ... 375
6.1 Preliminaries concerning explicit ODEs ... 375
6.2 Contractive DAEs and B-stable Runge–Kutta methods ... 378
6.3 Dissipativity ... 384
6.4 Lyapunov stability ... 387
6.5 Notes and references ... 394

Part III Computational aspects

7 Computational linear algebra aspects ... 399
7.1 Image and nullspace projectors ... 400

7.2 Matters of a properly stated leading term . . . 402
7.3 The basic step of the sequence . . . 404
7.3.1 Basis representation methods . . . 406
7.3.2 Basis representation methods—Regular case . . . 408
7.3.3 Projector representation method . . . 409
7.4 Matrix function sequences . . . 413
7.4.1 Stepping level by level . . . 413
7.4.2 Involved version for the regular case . . . 415
7.4.3 Computing characteristic values and index check . . . 416

8 Aspects of the numerical treatment of higher index DAEs . . . 419
8.1 Practical index calculation . . . 419
8.2 Consistent initialization . . . 424
8.3 Numerical integration . . . 426
8.4 Notes and references . . . 437

Part IV Advanced topics

9 Quasi-regular DAEs . . . 441
9.1 Quasi-proper leading terms . . . 441
9.2 Quasi-admissible matrix function sequences and admissible projector functions . . . 446
9.3 Quasi-regularity . . . 452
9.4 Linearization . . . 455
9.5 A DAE transferable into SCF is quasi-regular . . . 457
9.6 Decoupling of quasi-regular linear DAEs . . . 462
9.7 Difficulties arising with the use of subnullspaces . . . 468
9.8 Notes and references . . . 471
9.9 Hierarchy of quasi-admissible projector function sequences for general nonlinear DAEs . . . 475

10 Nonregular DAEs . . . 477
10.1 The scope of interpretations . . . 478
10.2 Linear DAEs . . . 482
10.2.1 Tractability index . . . 482
10.2.2 General decoupling . . . 486
10.2.2.1 G_μ has full column rank . . . 489
10.2.2.2 Tractability index 1, G_1 has a nontrivial nullspace . . . 493
10.2.2.3 Tractability index 2, G_2 has a nontrivial nullspace . . . 497
10.3 Underdetermined nonlinear DAEs . . . 499
10.4 Notes and references . . . 502

11 Minimization with constraints described by DAEs 505
11.1 Adjoint and self-adjoint DAEs 505
11.2 Extremal conditions and the optimality DAE 510
11.2.1 A necessary extremal condition and the optimality DAE 510
11.2.2 A particular sufficient extremal condition 520
11.3 Specification for controlled DAEs 522
11.4 Linear-quadratic optimal control and Riccati feedback solution 525
11.4.1 Sufficient and necessary extremal conditions 526
11.4.2 Riccati feedback solution 527
11.5 Notes and references 536

12 Abstract differential-algebraic equations 539
12.1 Index considerations for ADAEs 540
12.2 ADAE examples 544
12.2.1 Classical partial differential equations 545
12.2.1.1 Wave equation 545
12.2.1.2 Heat equation 546
12.2.2 A semi-explicit system with parabolic and elliptic parts 548
12.2.3 A coupled system of a PDE and Fredholm integral equations 552
12.2.4 A PDE and a DAE coupled by a restriction operator 554
12.3 Linear ADAEs with monotone operators 554
12.3.1 Basic functions and function spaces 555
12.3.2 Galerkin approach 558
12.3.3 Solvability 564
12.3.4 Continuous dependence on the data 571
12.3.5 Strong convergence of the Galerkin method 574
12.4 Notes and references 577

Appendices 581

A Linear algebra – basics 581
A.1 Projectors and subspaces 581
A.2 Generalized inverses 589
A.3 Parameter-dependent matrices and projectors 591
A.4 Variable subspaces 594

B Technical computations 599
B.1 Proof of Lemma 2.12 599
B.2 Proof of Lemma 2.41 605
B.3 Admissible projectors for $Nx' + x = r$ 612

C Analysis 627
C.1 A representation result 627

C.2 ODEs 629
C.3 Basics for evolution equations 632

References 637

Index 647

Notations

Abbreviations

ADAE	abstract DAE
BDF	backward differentiation formula
DAE	differential-algebraic equation
GLM	general linear method
IERODE	inherent explicit regular ODE
IESODE	inherent explicit singular ODE
IVP	initial value problem
MNA	modified nodal analysis
ODE	ordinary differential equation
PDAE	partial DAE
SCF	standard canonical form
SSCF	strong SCF

Common notation

$\mathbb{N}$	natural numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{K}$	alternatively $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{K}^n$	n-dimensional vector space
$M \in \mathbb{K}^{m,n}$	matrix with m rows and n columns
$M \in L(\mathbb{K}^n, \mathbb{K}^m)$	linear mapping from $\mathbb{K}^n$ into $\mathbb{K}^m$, also for $M \in \mathbb{K}^{m,n}$
$L(\mathbb{K}^m)$	shorthand for $L(\mathbb{K}^m, \mathbb{K}^m)$
M^{T}	transposed matrix
M^*	transposed matrix with real or complex conjugate entries
M^{-1}	inverse matrix
M^-	reflexive generalized inverse of M
M^+	Moore–Penrose inverse of M
$\ker M$	kernel of M, $\ker M = \{z \mid Mz = 0\}$

$\operatorname{im} M$	image of M, $\operatorname{im} M = \{z \mid z = My,\ y \in \mathbb{R}^n\}$
$\operatorname{ind} M$	index of M, $\operatorname{ind} M = \min\{k : \ker M^k = \ker M^{k+1}\}$
$\operatorname{rank} M$	rank of M
$\det M$	determinant of M
span	linear hull of a set of vectors
dim	dimension of a (sub)space
diag	diagonal matrix
$\{0\}$	set containing the zero element only
$M \cdot \mathcal{N}$	$= \{z \mid z = My,\ y \in \mathcal{N}\}$
$\forall$	for all
$\perp$	orthogonal set, $\mathcal{N}^\perp = \{z \mid \langle n, z\rangle = 0, \forall n \in \mathcal{N}\}$
$\otimes$	Kronecker product
$\oplus$	direct sum
$\ominus$	$\mathcal{X} = \mathcal{N}_i \ominus \mathcal{N}_j \Leftrightarrow \mathcal{N}_i = \mathcal{X} \oplus \mathcal{N}_j$
$\{A, B\}$	ordered pair
$\lvert \cdot \rvert$	vector and matrix norms in $\mathbb{R}^m$
$\lvert\lvert \cdot \rvert\rvert$	function norm
$\langle \cdot, \cdot \rangle$	scalar product in $\mathbb{K}^m$, dual pairing
$(\cdot \mid \cdot)_H$	scalar product in Hilbert space H
I, I_d	identity matrix (of dimension d)
$\mathcal{I}$	interval of independent variable
$(\ )'$	total time derivative, total derivative in jet variables
$(\ )_x$	(partial) derivative with respect to x
$\mathcal{C}(\mathcal{I}, \mathbb{R}^m)$	set of continuous functions
$\mathcal{C}^k(\mathcal{I}, \mathbb{R}^m)$	set of k-times continuously differentiable functions
$L_2(\mathcal{I}, \mathbb{R}^m)$	Lebesgue space
$H^1(\mathcal{I}, \mathbb{R}^m)$	Sobolev space

Special notation

$\mathcal{M}_0(t)$	obvious constraint
G_i	member of admissible matrix function sequence
r_i	$\operatorname{rank} G_i$, see Definition 1.17
S_j	$S_j = \ker \mathcal{W}_j B$, see Theorem 2.8 and following pages
N_j	$N_j = \ker G_j$, in Chapter 9: N_j subspace of $\ker G_j$
$\widehat{N_i}$	intersection: $N_0 + \cdots + N_{i-1} \cap N_i$, see (1.12)
N_{can}	canonical subspace, see Definition 2.36
$N_{can\,\mu}$	canonical subspace of an index μ DAE
S_{can}	canonical subspace (Definition 2.36)
$M_{can,q}$	set of consistent values, see (2.98)
$\mathcal{I}_{reg}$	set of regular points, see Definition 2.74
$X(\cdot, t_0)$	fundamental solution matrix normalized at t_0
dom_f	definition domain of f
$\mathcal{C}^k$-subspace	smooth subspace (cf. Section A.4)
$\mathcal{C}_*^\nu(\mathcal{G})$	set of reference functions, see Definition 3.17

$\mathcal{C}_D^1$	$\mathcal{C}_D^1(\mathcal{I},\mathbb{R}^m) := \{x \in \mathcal{C}(\mathcal{I},\mathbb{R}^m) : Dx \in \mathcal{C}^1(\mathcal{I},\mathbb{R}^n)\}$, see (1.78)
H_D^1	$H_D^1(\mathcal{I},\mathbb{R}^m) := \{x \in \mathcal{L}_2(\mathcal{I},\mathbb{R}^m) :\ Dx \in H^1(\mathcal{I},\mathbb{R}^n)\}$
$\mathcal{C}^{ind\ \mu}$	function space, see (2.104)
$\mathcal{G}$	regularity region

For projectors we usually apply the following notation:

Q	nullspace projector of a matrix G, $\operatorname{im} Q = \ker G$, $GQ = 0$
P	complementary projector, $P = I - Q$, $GP = G$
$\mathcal{W}$	projector along the image of G, $\ker \mathcal{W} = \operatorname{im} G$, $\mathcal{W}G = 0$
$P_i \cdots P_j$	ordered product, $\prod_{k=i}^{j} P_k$
Π_i	$\Pi_i := P_0 P_1 \cdots P_i$
Π_{can} $\Pi_{can\ \mu}$	canonical projector (of an index-μ DAE)
P_{dich}	dichotomic projector, see Definition 2.56

Introduction

Ordinary differential equations (ODEs) define relations concerning function values and derivative values of an unknown vector valued function in one real independent variable often called time and denoted by t. An explicit ODE

$$x'(t) = g(x(t),t)$$

displays the derivative value $x'(t)$ explicitly in terms of t and $x(t)$. An implicit ODE

$$f(x'(t),x(t),t) = 0$$

is said to be regular, if all its line-elements (x^1,x,t) are regular. A triple (x^1,x,t) belonging to the domain of interest is said to be a regular line-element of the ODE, if $f_{x^1}(x^1,x,t)$ is a nonsingular matrix, and otherwise a singular line-element. This means, in the case of a regular ODE, the derivative value $x'(t)$ is again fully determined in terms of t and $x(t)$, but in an implicit manner.

An ODE having a singular line-element is said to be a singular ODE. In turn, singular ODEs comprise quite different classes of equations. For instance, the linear ODE

$$tx'(t) - Mx(t) = 0$$

accommodates both regular line-elements for $t \neq 0$ and singular ones for $t = 0$. In contrast, the linear ODE

$$\begin{bmatrix} 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix} x'(t) + \begin{bmatrix} -\alpha & -1 & 0&0&0 \\ 0 & 1 & 0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix} x(t) - \begin{bmatrix} 0\\0\\0\\0\\ \gamma(t) \end{bmatrix} = 0 \tag{0.1}$$

has solely singular line-elements. A closer look at the solution flow of the last two ODEs shows a considerable disparity.

The ODE (0.1) serves as a prototype of a differential-algebraic equation (DAE). The related equation $f(x^1,x,t)=0$ determines the components x^1_1, x^1_3, x^1_4, and x^1_5 of x^1 in terms of x and t. The component x^1_2 is not at all given. In addition, there arises the consistency condition $x_5 - \gamma(t) = 0$ which restricts the flow.

DAEs constitute—in whatever form they are given—somehow uniformly singular ODEs: In common with all ODEs, they define relations concerning function values and derivative values of an unknown vector valued function in one real independent variable. However, in contrast to explicit ODEs, in DAEs these relations are implicit, and, in contrast to regular implicit ODEs, these relations determine just a part of the derivative values. A DAE is an implicit ODE which has solely singular line-elements.

The solutions of the special DAE (0.1) feature an ambivalent nature. On the one hand they are close to solutions of regular ODEs in the sense that they depend smoothly on consistent initial data. On the other hand, tiny changes of γ may yield monstrous variations of the solutions, and the solution varies discontinuously with respect to those changes. We refer to the figures in Example 1.5 to gain an impression of this ill-posed behavior.

The ambivalent nature of their solutions distinguishes DAE as being extraordinary to a certain extent.

DAEs began to attract significant research interest in applied and numerical mathematics in the early 1980s, no more than about three decades ago. In this relatively short time, DAEs have become a widely acknowledged tool to model processes subject to constraints, in order to simulate and to control these processes in various application fields.

The two traditional physical application areas, network simulation in electronics and the simulation of multibody mechanics, are repeatedly addressed in textbooks and surveys (e.g. [96, 25, 189]). Special monographs [194, 63, 188] and much work in numerical analysis are devoted to these particular problems. These two application areas and related fields in science and engineering can also be seen as the most important impetus to begin with systematic DAE research, since difficulties and failures in respective numerical simulations have provoked the analysis of these equations first.

The equations describing electrical networks have the form

$$A(d(x(t),t))' + b(x(t),t) = 0, \tag{0.2}$$

with a singular constant matrix A, whereas constrained multibody dynamics is described by equations showing the particular structure

$$x_1'(t) + b_1(x_1(t),x_2(t),x_3(t),t) = 0, \tag{0.3}$$

$$x_2'(t) + b_2(x_1(t),x_2(t),t) = 0, \tag{0.4}$$

$$b_3(x_2(t),t) = 0. \tag{0.5}$$

Those DAEs usually have large dimension. Multibody systems often comprise hundreds of equations and electric network systems even gather up to several millions of equations.

Many further physical systems are naturally described as DAEs, for instance, chemical process modeling, [209]. We agree with [189, p. 192] that DAEs arise probably more often than (regular) ODEs, and many of the well-known ODEs in application are actually DAEs that have been additionally explicitly reduced to ODE form.

Further DAEs arise in mathematics, in particular, as intermediate reduced models in singular perturbation theory, as extremal conditions in optimization and control, and by means of semidiscretization of partial differential equation systems.

Besides the traditional application fields, conducted by the generally increasing role of numerical simulation in science and technology, currently more and more new applications come along, in which different physical components are coupled via a network.

We believe that DAEs and their more abstract versions in infinite-dimensional spaces comprise *great potential for future mathematical modeling*. To an increasingly large extent, in applications, DAEs are automatically generated, often by coupling various subsystems, with large dimensions, but *without manifested mathematically useful structures*. Different modeling approaches may result in different kinds of DAEs. Automatic generation and coupling of various tools may yield quite opaque DAEs. Altogether, this produces the challenging task to *bring to light and to characterize the inherent mathematical structure of DAEs*, to provide test criteria such as index observers and eventually hints for creating better qualified model modifications. For a reliable practical treatment, which is the eventual aim, for numerical simulation, sensitivity analysis, optimization and control, and last but not least practical upgrading models, one needs pertinent information concerning the mathematical structure. Otherwise their procedures may fail or, so much the worse, generate wrong results. In consequence, providing practical assessment tools to uncover and to monitor mathematical DAE structures is one of the actual challenges. What are needed are criteria in terms of the original data of the given DAE. The projector based DAE analysis presented in this monograph is intended to address these questions.

Though DAEs have been popular among numerical analysts and in various application fields, so far they play only a marginal role in contiguous fields such as nonlinear analysis and dynamical systems. However, an input from those fields would be desirable. It seems, responsible for this shortage is the quite common view of DAEs as in essence nothing other than implicitly written regular ODEs or vector fields on manifolds, making some difficulties merely in numerical integration. The latter somehow biased opinion is still going strong. It is fortified by the fact that almost all approaches to DAEs suppose that the DAE is eventually reducible to an ODE as a basic principle. This opinion is summarized in [189, p. 191] as follows: *It is a fact, not a mere point of view, that a DAE eventually reduces to an ODE on a manifold. The attitude of acknowledging this fact from the outset leads to a reduc-*

tion procedure suitable for the investigation of many problems .... The mechanism of the geometric reduction procedure completely elucidates the "algebraic" and the "differential" aspects of a DAE. The algebraic part consists in the characterization of the manifold over which the DAE becomes an ODE and, of course, the differential part provides the reduced ODE. Also in [130] the explicit reduction of the general DAE

$$\mathfrak{f}(x'(t), x(t), t) = 0, \tag{0.6}$$

with a singular partial Jacobian $\mathfrak{f}_{x'}$, into a special reduced form plays a central role. Both monographs [189, 130] concentrate on related reduction procedures which naturally suppose higher partial derivatives of the function $\mathfrak{f}$, either to provide sequences of smooth (sub)manifolds or to utilize a so-called derivative array system. The differential geometric approach and the reduction procedures represent powerful tools to analyze and to solve DAEs. Having said that, we wonder about the misleading character of this purely geometric view, which underlines the closedness to regular ODEs, but loses sight of the ill-posed feature.

So far, most research concerning general DAEs is addressed to equation (0.6), and hence we call this equation a *DAE in standard form*. Usually, a solution is then supposed to be at least continuously differentiable.
In contrast, in the present monograph we investigate equations of the form

$$f((d(x(t), t))', x(t), t) = 0, \tag{0.7}$$

which show the derivative term involved by means of an extra function d. We see the network equation (0.2) as the antetype of this form. Also the system (0.3)–(0.5) has this form

$$\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \right)' + \begin{bmatrix} b_1(x_1(t), x_2(t), x_3(t), t) \\ b_2(x_1(t), x_2(t), t) \\ b_3(x_2(t), t) \end{bmatrix} = 0 \tag{0.8}$$

a priori. It appears that in applications actually DAEs in the form (0.7) arise, which precisely indicates the involved derivatives. The DAE form (0.7) is comfortable; it involves the derivative by the extra nonlinear function d, whereby $x(t) \in \mathbb{R}^m$ and $d(x(t), t) \in \mathbb{R}^n$ may have different sizes, as is the case in (0.8). A particular instance of DAEs (0.7) is given by the so-called *conservative form* DAEs [52]. Once again, the idea for version (0.7) originates from circuit simulation problems, in which this form is well approved (e.g. [75, 168]).

However, though equation (0.7) represents a more precise model, one often transforms it to standard form (0.6), which allows to apply results and tools from differential geometry, numerical ODE methods, and ODE software.

Turning from the model (0.7) to a standard form DAE one veils the explicit precise information concerning the derivative part. With this background, we are confronted with the question of what a DAE solution should be. Following the classical sense of differential equations, we ask for continuous functions being *as smooth as necessary*, which satisfy the DAE pointwise on the interval of interest. This is a common understanding. However, there are different opinions on the meaning of the appropriate smoothness. Having regular ODEs in mind one considers continuously differentiable functions $x(\cdot)$ to be the right candidates for solutions. Up to now, most DAE researchers adopt this understanding of the solution which is supported by the standard DAE formulation. Furthermore, intending to apply formal integrability concepts, differential geometry and derivative array approaches one is led to yet another higher smoothness requirement. In contrast, the multibody system (0.8) suggests, as solutions, continuous functions $x(\cdot)$ having just continuously differentiable components $x_1(\cdot)$ and $x_2(\cdot)$.

An extra matrix figuring out the derivative term was already used much earlier (e.g. [153, 152, 154]); however, this approach did not win much recognition at that time. Instead, the following interpretation of standard form DAEs (e.g. [96]) has been accepted to a larger extent: Assuming the nullspace of the partial Jacobian $\mathfrak{f}_{x'}(x',x,t)$ associated with the standard form DAE (0.6) to be a $\mathcal{C}^1$-subspace, and to be independent of the variables x' and x, one interprets the standard form DAE (0.6) as a short description of the equation

$$\mathfrak{f}((P(t)x(t))' - P'(t)x(t), x(t), t) = 0, \tag{0.9}$$

whereby $P(\cdot)$ denotes any continuously differentiable projector valued function such that the nullspaces $\ker P(\cdot)$ and $\ker \mathfrak{f}_{x'}(x',x,\cdot)$ coincide. This approach is aligned with continuous solutions $x(\cdot)$ having just continuously differentiable products $(Px)(\cdot)$. Most applications yield even constant nullspaces $\ker \mathfrak{f}_{x'}$, and hence constant projector functions P as well. In particular, this is the case for the network equations (0.2) and the multibody systems (0.8).

In general, for a DAE given in the form (0.7), a solution $x(\cdot)$ should be a continuous function such that the superposition $u(\cdot) := d(x(\cdot),\cdot)$ is continuously differentiable. For the particular system (0.8) this means that the components $x_1(\cdot)$ and $x_2(\cdot)$ are continuously differentiable, whereas one accepts a continuous $x_3(\cdot)$.

The question in which way the data functions f and d should be related to each other leads to the notions of *DAEs with properly stated leading term or properly involved derivative*, but also to *DAEs with quasi-proper leading term*. During the last 15 years, the idea of using an extra function housing the derivative part within a DAE has been emphatically pursued. This discussion amounts to the content of this monograph. Formulating DAEs with properly stated leading term yields, in particular, symmetries of linear DAEs and their adjoints, and further favorable consequences concerning optimization problems with DAE constraints. Not surprisingly, numerical discretization methods may perform better than for standard form DAEs. And last, but not least, this approach allows for appropriate generalizations to apply to abstract differential- algebraic systems in Hilbert spaces enclosing PDAEs. We think that, right from the design or modeling stage, it makes sense to look for properly involved derivatives.

This monograph comprises an elaborate analysis of DAEs (0.7), which is accompanied by the consideration of essential numerical aspects. We regard DAEs from an analytical point of view, rather than from a geometric one. Our main objective consists in the structural and qualitative characterization of DAEs as they are given a priori, without supposing any knowledge concerning solutions and constraints. Afterwards, having the required knowledge of the DAE structure, also solvability assertions follow. Only then do we access the constraints. In contrast, other approaches concede full priority of providing constraints and solutions, as well as transformations into a special form, which amounts to solving the DAE.

We believe in the great potential of our concept in view of the further analysis of classical DAEs and their extensions to abstract DAEs in function spaces. We do not at all apply derivative arrays and prolongated systems, which are commonly used in DAE theory. Instead, certain admissible matrix function sequences and smartly chosen admissible projector functions formed only from the first partial derivatives of the given data function play their role as basic tools. Thereby, continuity properties of projector functions depending on several variables play their role, which is not given if one works instead with basises. All in all, this allows an analysis on a low smoothness level. We pursue a fundamentally alternative approach and present the first rigorous structural analysis of general DAEs in their originally given form without the use of derivative arrays, without supposing any knowledge concerning constraints and solutions.

The concept of a projector based analysis of general DAEs was sketched first in [160, 171, 48], but it has taken its time to mature. Now we come up with a unique general theory capturing constant coefficient linear problems, variable coefficient linear problems and fully nonlinear problems in a hierarchic way. We address a further generalization to abstract DAEs. It seems, after having climbed the (at times seemingly pathless) mountain of projectors, we are given transparency and beautiful convenience. By now the projector based analysis is approved to be a prospective way to investigate DAEs and also to yield reasonable open questions for future research.

The central idea of the present monograph consists in a rigorous definition of regularity of a DAE, accompanied with certain characteristic values including the tractability index, which is related to an open subset of the definition domain of the data function f, a so-called *regularity region*. Regularity is shown to be stable with respect to perturbations. Close relations of regularity regions and linearizations are proved. In general, one has to expect that the definition domain of f decomposes into several regularity regions whose borders consist of critical points. Solutions do not necessarily stay in one of these regions; solutions may cross the borders and undergo bifurcation, etc.

The larger part of the presented material is new and as yet unpublished. Parts were earlier published in journals, and just the regular linear DAE framework (also critical points in this context) is available in the book [194].

The following basic types of DAEs can reasonably be discerned:

✓ fully implicit nonlinear DAE with nonlinear derivative term

$$f((d(x(t),t))',x(t),t)=0, \tag{0.10}$$

✓ fully implicit nonlinear DAE with linear derivative term

$$f((D(t)x(t))', x(t), t) = 0, \tag{0.11}$$

✓ quasi-linear DAE with nonlinear derivative term (involved linearly)

$$A(x(t), t)(d(x(t), t))' + b(x(t), t) = 0, \tag{0.12}$$

✓ quasi-linear DAE with linear derivative term

$$A(x(t), t)(D(t)x(t))' + b(x(t), t) = 0, \tag{0.13}$$

✓ linear DAE with variable coefficients

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \tag{0.14}$$

✓ linear DAE with constant coefficients

$$A(Dx(t))' + Bx(t) = q(t), \tag{0.15}$$

✓ semi-implicit DAE with explicitly given derivative-free equation

$$f_1((d(x(t), t))', x(t), t) = 0, \tag{0.16}$$
$$f_2(x(t), t) = 0, \tag{0.17}$$

✓ semi-implicit DAE with explicitly partitioned variable and explicitly given derivative-free equation

$$f_1(x_1'(t), x_1(t), x_2(t), t) = 0, \tag{0.18}$$
$$f_2(x_1(t), x_2(t), t) = 0, \tag{0.19}$$

✓ semi-explicit DAE with explicitly partitioned variable and explicitly given derivative-free equation

$$x_1'(t) + b_1(x_1(t), x_2(t), t) = 0, \tag{0.20}$$
$$b_2(x_1(t), x_2(t), t) = 0. \tag{0.21}$$

So-called Hessenberg form DAEs of size r, which are described in Section 3.5, form further subclasses of semi-explicit DAEs. For instance, the DAE (0.8) has Hessenberg form of size 3. Note that much work developed to treat higher index DAEs is actually limited to Hessenberg form DAEs of size 2 or 3.

The presentation is divided into Part I to Part IV followed by Appendices A, B, and C.

Part I describes the core of the projector based DAE analysis: the construction of admissible matrix function sequences associated by admissible projector functions and the notion of regularity regions.

Chapter 1 deals with constant coefficient DAEs and matrix pencils only. We reconsider algebraic features and introduce into the projector framework. This can be skipped by readers familiar with the basic linear algebra including projectors.

The more extensive Chapter 2 provides the reader with admissible matrix function sequences and the resulting constructive projector based decouplings. With this background, a comprehensive linear theory is developed, including qualitative flow characterizations of regular DAEs, the rigorous description of admissible excitations, and also relations to several canonical forms and the strangeness index.

Chapter 3 contains the main constructions and assertions concerning general regular nonlinear DAEs, in particular the regularity regions and the practically important theorem concerning linearizations. It is recommended to take a look to Chapter 2 before reading Chapter 3.

We emphasize the hierarchical organization of Part I. The admissible matrix function sequences built for the nonlinear DAE (0.10) generalize those for the linear DAE (0.14) with variable coefficients, which, in turn, represent a generalization of the matrix sequences made for constant coefficient DAEs (0.15).

Part IV continues the hierarchy in view of different further aspects. Chapter 9 about quasi-regular DAEs (0.10) incorporates a generalization which relaxes the constant-rank conditions supporting admissible matrix function sequences. Chapter 10 on nonregular DAEs (0.11) allows a different number of equations and of unknown components. Finally, in Chapter 12, we describe abstract DAEs in infinite-dimensional spaces and include PDAEs.

Part IV contains the additional Chapter 11 conveying results on minimization with DAE constraints obtained by means of the projector based technique.

Part II is a self-contained index-1 script. It comprises in its three chapters the analysis of regular index-1 DAEs (0.11) and their numerical integration, addressing also stability topics such as contractivity and stability in Lyapunov's sense. Part II constitutes in essence an up-to-date improved and completed version of the early book [96]. While the latter is devoted to standard form DAEs via the interpretation (0.9), now the more general equations (0.11) are addressed.

Part III adheres to Part I giving an elaborate account of computational methods concerning the practical construction of projectors and that of admissible projector functions in Chapter 7. A second chapter discusses several aspects of the numerical treatment of regular higher index DAEs such as consistent initialization and numerical integration.

Appendix B contains technically involved costly proofs. Appendices A and C collect and provide basic material concerning linear algebra and analysis, for instance the frequently used $\mathcal{C}^1$-subspaces.

Plenty of reproducible small academic examples are integrated into the explanations for easier reading, illustrating and confirming the features under consideration. To this end, we emphasize that those examples are always too simple. They bring to light special features, but they do not really reflect the complexity of DAEs.

The material of this monograph is much too comprehensive to be taught in a standard graduate course. However different combinations of selected chapters should be well suited for those courses. In particular, we recommend the following:

- Projector based DAE analysis (Part I, possibly without Chapter 1).
- Analysis of index-1 DAEs and their numerical treatment (Part II, possibly plus Chapter 8).
- Matrix pencils, theoretical and practical decouplings (Chapters 1 and 7).
- General linear DAEs (Chapter 2, material on the linear DAEs of Chapters 10 and 9).

Advanced courses communicating Chapter 12 or Chapter 11 could be given to students well grounded in DAE basics (Parts I and II) and partial differential equations, respectively optimization.