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Extremal Polynomials and Riemann Surfaces

Translated from Russian by Nikolai Kruzhilin

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To the blessed memory of Nikolai Sergeevich Bakhvalov

Contents

Notations

$\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$	The sets of complex, real, rational, integer, and positive integer numbers, respectively.
$\hat{\mathbb{R}}$	The extended real line, a circle.
$\mathbb{CP}^1 = \hat{\mathbb{C}}$	The Riemann sphere.
$\mathbb{H} := \{x \in \mathbb{C}\colon \mathrm{Im}(x) > 0\}$	The open upper half-plane.
$\#\{\dots\}$	The cardinal number of a set.
$\square$	The end of a proof.
$M(\mathbf{e})$	The real hyperelliptic curve with branch divisor **e**.
$\mathbf{e} := \{e_s\}_{s=1}^{2g+2}$	An (unordered) system of $g - k + 1$ pairs of complex conjugate points and $2k$ real points.
∞_-, ∞_+	Two points on the curve M lying over the point at infinity.
J	The hyperelliptic involution of M.
$\bar{J}$	The anticonformal involution (reflection) on M.
C_s^+, C_s^-	Even and odd 1-cycles on M, respectively.
$H_1^-(M, \mathbb{Z})$	The lattice of odd integral 1-cycles on M (of rank $g + 1$).
L_M	The sublattice of $H_1^-(M, \mathbb{Z})$.
$\langle C^* \vert C \rangle$	The value of a functional (cocycle) C^* at a cycle C.
η_M	The (real) abelian differential of the 3d kind assigned to M, with simple poles at infinity, residues ± 1, and purely imaginary periods.
$\mathfrak{A}_1^+$	The group of affine orientation-preserving motions of the real line.
Br_m	The pure braid group on m strands.
$\mathscr{H}_g^k$	The moduli space of genus g curves with k real ovals.
$\tilde{\mathscr{H}}_g^k \cong \mathbb{R}^{2g}$	The universal covering space of the moduli space.
QC	The group of quasiconformal homeomorphisms of the upper half-plane $\mathbb{H}$ fixing the points -1, $+1$, and ∞.

$\mathsf{QC}(\mathbf{e})$	The subgroup of homeomorphisms stabilizing the divisor **e**.
$\mathsf{QC}^0(\mathbf{e})$	The identity component of $\mathsf{QC}(\mathbf{e})$.
$\mathscr{T}_g^k$	The Teichmüller space of the disc with $g-k+1$ punctures and $2k+1$ marked points on the boundary.
$\mathrm{Mod}(\mathbf{e})$	The modular group of the punctured half-plane $\mathbb{H} \setminus \mathbf{e}$.
$\mathfrak{G}$	The free product of $g+1$ rank-two groups with generators G_s, $s = 0, 1, \dots, g$; also a realization of this abstract group by a Kleinian group.
$\mathrm{fix}\, G_s$	The fixed point set of the linear fractional map $G_s(u)$.
$\mathfrak{S}$	The Schottky group with generators $S_l := G_l G_0$, an index-two subgroup of $\mathfrak{G}$.
$\mathscr{D}(\mathfrak{G})$	The domain of discontinuity of the Kleinian group $\mathfrak{G}$.
$\Lambda(\mathfrak{G})$	The limit set of the group $\mathfrak{G}$.
$\mathscr{G}_g^k$	The deformation space of the special Kleinian group $\mathfrak{G}$.
$\mathfrak{g} = \{G_s\}_{s=0}^g$	An element of the deformation space: an ordered set of generators G_s of the group $\mathfrak{G}$.
$\{c_s, r_s\}_{s=1}^g$	A global system of coordinates in $\mathscr{G}_g^k$ related to the parameters of linear second-order rotations G_s.
$\mathsf{R}(\mathfrak{g})$	A fundamental domain of the Kleinian group $\mathfrak{G}$ generated by the system $\mathfrak{g}$ of linear fractional transformations, which is bounded by the imaginary axis and circles $C_1, C_2, \dots, C_g$.
$\mu(x)$, $\mu(x)\overline{dx}/dx$	A Beltrami coefficient and a Beltrami differential, respectively.
$\mathscr{L}_g^k$	A labyrinth space, a model for the universal covering space $\tilde{\mathscr{H}}_g^k$.
Λ	A labyrinth, a special system of $g+1$ cuts connecting pairwise points in the branch divisor **e**.
$H_1\mathscr{H}_g^k$	The vector bundle of homology spaces over the moduli space.
$H_1^-\mathscr{H}_g^k$	The subbundle of odd 1-homology.
$\Omega^1\mathscr{H}_g^k$	The vector bundle of real abelian differentials with simple poles at infinity.
$\Pi\colon \Omega^1\tilde{\mathscr{H}}_g^k \to \mathbb{R}^{2g+1}$	The global period map.
$\Pi_-\colon \tilde{\mathscr{H}}_g^k \to \mathbb{R}^g$	The restriction of the global period map to the manifold of distinguished differentials η_M on curves M.
$W(x) := \left\| \mathrm{Re} \int_{(e_s,0)}^{(x,w)} \eta_M \right\|$	A globally defined (Green's) function on the sphere.
Γ	The finite tree related to the foliation $(\eta_M)^2 > 0$.
$\Gamma_\vert$	A part of the graph Γ, the zero set of the function $W(x)$.

$\Gamma_\bullet^0 := \Gamma_\bullet \cap \mathbb{R}$	$\Gamma_+^0 := \Gamma_\bullet \cap \mathbb{H}$, here $\bullet = \vert, -$ or empty.
$\mathrm{Comb}\{\Gamma\}$	A comb-like domain constructed from a weighted graph.
$\sigma(x)$	A cutoff function equal to 1 in a 2-dimensional neighbourhood of the point $x = 0$.
$\varkappa(t)$	A "Courant tent", a function of the real variable t.
$\theta(t)$	The Heavyside function of the real variable t.
$\mathbb{T}(C_*)$	A fibre of the period map Π_- over the point C_* in the space of functionals.
$\mathfrak{i}$	A $(g - k + 1)$-subset of $\{1, 2, \ldots, g\}$.
$\kappa(\mathfrak{i})$	The braid on $g + 2$ strands corresponding to $\mathfrak{i}$.
R	A fundamental domain of the Schottky group $\mathfrak{S}$ in Chap. 6 bounded by $2g$ circles $-C_g, \ldots, -C_2, -C_1, C_1, C_2, \ldots, C_g$.
$(u, u'; z, z')$	A Schottky function, the exponential of an abelian integral of the 3d kind with poles z and z' over a curve from u to u'.
$E_s(u)$	The exponent of an abelian integral of the 1st kind taken from ∞ to u.
E_{sl}	The Schottky constants, the exponentials of the period matrix of the curve.
$\mathsf{M}(u) := \begin{Vmatrix} -u & u^2 \\ -1 & u \end{Vmatrix}$	The Hejhal matrix.
$\mathrm{diam}(\cdot)$	The Euclidean diameter of a set.
$\mathrm{dist}(\cdot, \cdot)$	The Euclidean distance between points or sets.
D_u^l	The lth partial derivative with respect to u.
C_+, C_-, C_0, C_1	Four distinguished cycles on a curve $M \in \mathscr{H}_2^1$.
$E_+(u)$, $E_-(u)$	The Schottky functions in Chap. 7.
E_{++}, E_{+-}, E_{-+}, E_{--}	The Schottky constants in Chap. 7, the exponentials of entries of the period matrix of the curve.

Introduction

A couple of years before the Crimean War broke out, P. L. Chebyshev travelled to Great Britain to get to know the most advanced technologies of that time. On his return to Russia, he concentrated on a purely engineering problem of minimizing the friction in the joins of Watt's parallelogram which turns the back–and–forth motion of the steam engine into wheel rotation. Chebyshev's investigations led to the eventual replacement of the parallelogram linkage by the crankshaft, which is still in use today. As a by–product of the evolution of technology, the *Chebyshev polynomials* were discovered, *un miracle d'analyse*, in J. Bertrand's words, which since then have found their way into all the textbooks. These polynomials have turned out to solve the simplest problems of constrained minimization of the *deviation*

$$\|P_n\|_E := \max_{x \in E} |P_n(x)|, \tag{1}$$

where E is a compact subset of the real axis, over the space of real polynomials

$$\left\{ P_n(x) = \sum_{s=0}^{n} c_s x^s \right\} \cong \mathbb{R}^{n+1}. \tag{2}$$

Now, 150 years later, steam engines are no longer used, but an interest into least deviation problem is still here [33, 146]. Today it is connected, for example, with optimizing numerical algorithms [97,114] and signal processing [14,45]. We present several typical problems.

Problem A. *Let E be a system of several finite intervals on the real axis. Minimize the norm $\|P_n\|_E$ of a polynomial satisfying fixed linear constraints on its coefficients $c_0, c_1, \dots, c_n$.*

The least deviation polynomial with fixed leading coefficient is called the *Chebyshev polynomial* on E. The Zolotarëv problem [146, 160] corresponds to the case of one interval $E = [-1, 1]$ and several fixed leading coefficients of the polynomial. The V. A. Markov problem [101] corresponds to one interval E and one linear constraint.

Problem B. *Find a polynomial $R_n(x)$ approximating the exponential function to order $p \le n$ at $x = 0$, $R_n(x) = 1 + x + x^2/2! + \cdots + x^p/p! + o(x^p)$, such that the deviation $\|R_n\|_E$ does not exceed* 1 *on the largest possible interval $E = [-L, 0]$, $L > 0$.*

The problem of *optimal stability polynomial $R_n(x)$* was stated by several authors [65, 71, 105, 127, 142] in the late 1950s/early 1960s, in connection with designing explicit n–stage stable Runge–Kutta methods of accuracy order p.

Solving such extremal problems numerically for practically interesting degrees $n \approx 1{,}000$ is well known to be very complicated. The algorithms due to Remez [77, 98, 126], Lebedev [95], Peherstorfer–Schiefermayr [118], or convex programming methods [129, 146] require a large amount of computational resources for the following reasons: (1) the solution is sought by iterations in a high-dimensional space (of dimension of order n) and (2) the norm of a polynomial is a non-smooth function of its coefficients which is difficult to evaluate.

The classical approach, when a solution is expressed by an explicit formula, is free from these deficiencies [122]. One hundred and fifty years ago, when no computers were known, iterative methods of solution were considered unsatisfactory. The first least deviation problems were solved by producing analytic expressions for the polynomial and its argument, thus defining the polynomial parametrically:

$$T_n(u) := \cos(nu); \quad x(u) := \cos(u), \quad u \in \mathbb{C} \tag{3}$$

(Chebyshev [47]) and

$$Z_n(u) := \frac{1}{2}\left\{\left[\frac{H(a+u)}{H(a-u)}\right]^n + \left[\frac{H(a-u)}{H(a+u)}\right]^n\right\}; \tag{4}$$

$$x(u) := \frac{\mathrm{sn}^2(u) + \mathrm{sn}^2(a)}{\mathrm{sn}^2(u) - \mathrm{sn}^2(a)}, \quad u \in \mathbb{C},$$

(Zolotarëv [160]). In the last formula, $H(\cdot)$ is the elliptic theta function with modulus $k \in (0,1)$ in Jacobi's (outdated) notation [155], $\mathrm{sn}(\cdot)$ is the elliptic sine function with the same modulus, and $a := mK(k)/n$, $m = 1, 2, \dots, n-1$, is a phase shift, where $K(k)$ is the complete elliptic integral with the same modulus k. These parametric formulae can be treated as follows: the function $x(u)$ is automorphic with respect to the discontinuous action of some group $\mathfrak{G}$ on the complex plane. The orbit manifold $\mathbb{C}/\mathfrak{G}$ is the Riemann sphere (in the case (3)) or a torus (in the case (4)). The expressions for $T_n(u)$ and $Z_n(u)$ are well defined on the corresponding quotient spaces and are degree n polynomials of x. We see that classical solutions are related to algebraic curves of small genus $g = 0, 1$, and the complexity of their computation it independent of the degree n of the polynomial.

Developing the classical approach to problems of least deviation in the uniform norm, instead of the full space of polynomials (2), we shall seek the solution on certain low-dimensional submanifolds of this space. The *alternation principle* discovered by Chebyshev [33, 146] and subsequently explained by convex analysis says that the following situation is typical. *Most critical points of the solution $T(x)$ are simple, correspond to the values $\pm\|T(x)\|_E$, and lie in E.* Polynomials of this kind are very special; they fill low-dimensional submanifolds of the space (2). Here is a geometric explanation for this. A solution to an extremal problem corresponds to a tangency between the (usually linear) submanifold of (2) corresponding to the constraints of the problem and the sphere formed by the polynomials of equal norm. A ball corresponding to the uniform norm is a convex curvilinear polytope: its boundary is not smooth and is partitioned into faces of various dimension. Low-dimensional faces are more protruding, so it is little surprising that planes touch these faces more often. (For example, the corners of an old suitcase are worn out the most; a pencil falling on the floor reaches it with its tip more often than flatwise, etc.) On the other hand, higher-dimensional faces of a ball are ruled and their contact with linear subspaces can occur along continua: then the corresponding minimum problem is not uniquely solvable. We shall show that for polynomials solving least deviation problems the above–described form is more common. This justifies the following definition.

Definition 1. A real polynomial $P(x)$ is called a (*normalized*) *g–extremal polynomial* if all of its critical points, apart from g points, are simple and the corresponding values of the polynomial are ± 1.

Here the parameter g, the number of exceptional critical points, can be calculated by the formula

$$g = \sum_{x:P(x)\neq\pm 1} \operatorname{ord} P'(x) + \sum_{x:P(x)=\pm 1} \left[\frac{1}{2} \operatorname{ord} P'(x)\right], \tag{5}$$

where $\operatorname{ord} P'(x)$ is the order of the zero of the derivative of P at the point $x \in \mathbb{C}$ and $[\cdot]$ is the integer part of a number.

The term "extremality" is used here for two reasons. On the one hand we distinguish the critical points which do not obey certain general rules. On the other, the polynomials with small value of g are more likely to solve various extremal problems involving the uniform norm. They are important for applications and, slightly abusing the language, we shall call them simply *extremal* polynomials. Polynomials with extremality parameters $g = 0$ and $g = 1$ were discovered 150 years ago and are known as Chebyshev and Zolotarëv polynomials, respectively. We give the graphs of several 2-extremal polynomials in Fig. 6.4.

Our aim in this book is to investigate g–extremal polynomials and to use them for an effective solution of optimization problems. The ideas behind our approach to problems of least deviation in the uniform norm and the technical realization of this approach are more complicated than the algorithms due to Remez, Lebedev, and other authors mentioned above. However it has an advantage: the complexity of computing a solution using explicit analytic formulae does not depend on the degree n of the polynomial, as we clearly see in the classical Chebyshev and Zolotarëv formulae. On the other hand, the bulk of calculations grows rapidly with the parameter g, so the natural range of application for this method is the case when solutions have a high degree n, but only a few of constraints are imposed on their coefficients and the set E consists of a few components.

Polynomials, as well as rational and algebraic functions with few critical values, is a classical object of mathematical investigations, lying on the border between "continuous" and "discrete" mathematics.

One of the lines of these investigations goes back to Hurwitz [78]; it is related to classifying branched covers of the sphere, investigating the strata of the corresponding discriminant set, Lyashko–Loijenga maps, Belyi pairs, and Grothendieck's *dessins d'enfants*. In recent years this approach has been extensively developed by the Moscow Mathematical School (see e.g. the comments and references to Problem 1970-15 in "Arnold's Problems" [15] and also [87, 90, 91, 113, 158, 163]). For instance, in [136] polynomials with precisely two finite critical values (Shabat polynomials) and their applications to number theory are considered.

Another line of research dates back to Chebyshev [47] or, in fact, to Niels Abel [2]. It is related to the investigation of Pell's equation[1] with polynomial coefficient, continued fraction expansions, and conditions for the reduction of abelian integrals, when these turn into integrals of lower genus and, in particular, can be reduced to elementary functions [19]. For a survey of this line of research we refer the reader to [139], [144] and also to [9, 13, 46, 86]. A characteristic feature of this second approach is effective calculations and links to applications. Bearing all this in mind, we take the second approach and wish to develop it to the level of effective numerical calculations [37, 40, 43].

Chebyshev's programme. In [47] Chebyshev showed that solutions $P(x)$ of the minimax problems that he stated satisfy Pell's equation

$$P^2(x) - D(x)Q^2(x) = 1 \tag{6}$$

with a square–free polynomial $D(x) := \prod_{s=1}^{2g+2}(x - e_s)$ determined by the data of the optimization problem. The first author to consider Pell's equation with a polynomial coefficient $D(x)$ was Abel [2], who proposed two tests for its solvability: (1) the function $\sqrt{D(x)}$ expands in a periodic continued fraction; (2) for some choice of the coefficients c_s the primitive

$$\int \frac{x^g + \sum_{s=0}^{g-1} c_s x^s}{\sqrt{D(x)}} dx$$

can be expressed "in terms of logarithms" [46]. Chebyshev proposed to seek solutions to (6) as the cosines of hyperelliptic integrals. Here is his argument:

[1]Pell's equation is the Diophantine equation $P^2 - DQ^2 = 1$, where D is a fixed square–free integer coefficient and P and Q are unknown integers. It was considered by William Brounker (1657), Pierre Fermat, John Wallis. By confusion Euler related it to John Pell [58].

differentiating Pell's equation we easily see that the polynomial $P'(x)$ is a multiple of $Q(x)$, so that $P'(x) = n\Big(x^g + \sum_{s=0}^{g-1} c_s x^s\Big)Q(x)$, $n := \deg P$, with some coefficients c_s. Now substituting for $Q(x)$ its expression from (6) we obtain *Chebyshev's differential equation*

$$\frac{P'(x)}{\sqrt{P^2(x)-1}} = n\frac{x^g + \sum_{s=0}^{g-1} c_s x^s}{\sqrt{D(x)}}, \quad n := \deg P.$$

Integrating it we readily obtain a solution to Pell's equation:

$$P(t) = \cos\left(in\int_{e_j}^{t} \frac{x^g + \sum_{s=0}^{g-1} c_s x^s}{\sqrt{D(x)}} dx \right);$$

in what follows we call this the Chebyshev representation for the solution. The function in the right hand side of this formula is a polynomial if and only if one of Abel's criteria holds.[2] In [47] Chebyshev criticized Abel's criteria as being not sufficiently effective, while here he asks *how, given a coefficient $D(x)$ of Pell's equation, can we understand whether* (6) *is solvable and, if the answer is "yes", how we can effectively find the solutions*.

For elliptic integrals this research programme was outlined by Chebyshev and was completely carried out by his student E. I. Zolotarëv in 1868–1877 [160–162].

The history of Zolotarëv's heritage is of interest. In 1872 he was attending lectures of Weierstrass in Berlin and communicated his achievements. This work revived Weierstrass's interest in the reductions of abelian integrals which eventually resulted in Weierstrass-Poincare reduction theory [119]. In the 1930s Zolotarëv's works were still known in Germany and were applied by W. Cauer [44, 45] in electrical engineering. However, his contributions were later all but forgotten in the mathematical community, although some of them were repeatedly discovered anew. In the 1990s, thanks to enthusiastic efforts of several authors, the first among these being Todd [147] and Lebedev [96], Zolotarëv's "priority right" was restored. Surprisingly, Zolotarëv's name remained known to the community of electrical engineers all the time.

The next significant step in implementing Chebyshev's programme was made by N. I. Akhiezer, who used in these problems the language of geometric function theory. In 1928, for the solution of the Zolotarëv problem with three fixed coefficients, Akhiezer put forward an Ansatz involving Schottky functions of

[2] Alternatively, all the periods of the abelian integral in the last formula must be integer multiples of 2π, the period of cos. This reflects the discrete aspect of the problem under consideration.

curves of genus $g = 2$. His work [9] was far ahead of his time, although his solution was incomplete: for instance, he could not tell whether the system of (Abel's) transcendental equations for the parameters of the substitution was solvable. Akhiezer's methods used the machinery of Green's functions in the plane cut in some special way, which shaded the connections with algebraic curves. Unfortunately, in his further papers [10, 12] on approximation theory Akhiezer limited himself to elliptic functions, which led to Zolotarëv polynomials and fractions.

Elliptic integrals were a subject of interest for researchers as long ago as the second half of the seventeenth century (John Wallis and the brothers Jacob and Johann Bernoulli). Elliptic integrals gave name to some of the simplest Riemann surfaces, elliptic curves. The theory of elliptic functions called by Klein the *heart and soul of mathematics of nineteenth century* [83], has been extensively developed for more than 250 years, and the literature devoted to it includes tens of thousands of publications. Numerical algorithms have been designed for an effective treatment of curves of genus $g = 1$ and now are implemented in modern computer software. This is one reason why the interest in elliptic functions has revived in recent decades. The interest has arisen in the theory of extremal and orthogonal polynomials [96, 112, 116], within the algebro–geometric approach to integrable systems and scattering by double–periodic potentials [59, 67, 68], and also in the complex geometric theory of one–dimensional integral equations [36, 39].

Riemann surfaces were introduced by Riemann in 1851 as ramified covers of the sphere. The basics of their theory were established by such German mathematicians as Jacobi, Weierstrass, Max Noether, Klein, Hurwitz, Fricke, Koebe, Weyl, and Teichmüller. The theory of Riemann surfaces has beautiful applications in mathematics (embeddings of minimal surfaces, optimization of numerical algorithms), theoretical and mathematical physics (conformal field theory, string theory, finite gap integration, matrix models), industry (electrical filters, encoding), and even medicine (parametrization of the brain surface). The large amount of knowledge about algebraic curves and their deformation spaces makes it possible to use these objects in calculations. Apparently, the first computer evaluation of special functions related to higher genus curves was also related to modelling non-linear waves in the mid–1980s [35]. Today effective computation of function–theoretic objects like abelian integrals, differentials, spinors, Riemann thetas, etc. on higher genus Riemann surfaces is a vibrant branch of numerical analysis with research groups all over the world (TU Berlin [22, 131], Imperial College (London) [50–52], University of Washington [53–55], Florida State University [30, 75, 76, 133, 134] to mention just a few).

Numerical analysis of Riemann surfaces and their moduli spaces is based on the use of Riemann theta functions [56, 61, 108] or Schottky functions [18, 27, 29, 132]. The second way is slightly easier because it allows us to avoid solving numerically the notorious Schottky problem of characterizing the period matrices of Riemann surfaces. In the context of optimization problems for polynomials this approach was put forward by Akhiezer [9] and developed further later on to yield numerical results [37, 41, 43]. It leans on a theorem [35] stating that *real* algebraic curves can be uniformized by some special Schottky groups $\mathfrak{S}$, whose linear Poincaré theta series converge absolutely and uniformly on compact subsets of the domain of discontinuity of the group. This result fails for general Schottky groups: Poincaré even believed that linear series were never convergent (see the history of this issue and a survey of results in [7, 8, 110, 145]).

The reader's background. In this book we use methods from various areas of mathematics. The reader is assumed to be familiar with basic complex analysis [92], the theory of Riemann surfaces [60, 107, 123, 141], quasiconformal maps [3, 152] and geometry of discrete groups [32, 109]. Some background in Strebel foliations [143] and Teichmüller theory [4, 20] is also advisable (although not necessary).

The area of application. The following algorithms of numerical mathematics can be optimized with the use of our results here: (1) designing explicit stable difference schemes for ordinary differential equations [71]; (2) Chebyshev acceleration in the iterative solution of large systems of linear equations with a non-singular symmetric matrix. Our approaches can be used for optimizing electric schemes and electronic filters. Our method for calculating special functions connected with Riemann surfaces can be used for numerical simulation in conformal field theory and finite–gap integration.

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A Summary

Chapter 1 is concerned with our definition of extremal polynomials. We present examples of optimization of numerical algorithms leading to minimax problems of Chebyshev type. Then extremal problems are investigated with the use of convex analysis. A solvability criterion for least deviation problems with several linear constraints is established, which generalizes the classical alternance principle. The shares of polynomials of different types among solutions of problems with fixed number of constraints are evaluated: the most typical solutions are polynomials which we call extremal, with small extremality parameter g. Properties of optimal stability polynomial are investigated.

Chapter 2. The representation for extremal polynomials discussed in this chapter goes back to Chebyshev and is a geometric interpretation of Pell's equation with a polynomial coefficient. With each polynomial $P(x)$ we associate a hyperelliptic curve

$$M = M(\mathbf{e}) = \left\{(x, w) \in \mathbb{C}^2 \colon w^2 = \prod_{s=1}^{2g+2} (x - e_s)\right\}, \tag{7}$$

with branch divisor $\mathbf{e} := \{e_s\}_{s=1}^{2g+2}$ consisting of the zeros of odd multiplicity of $P^2(x) - 1$. The genus g of this curve is equal to the number of exceptional critical points of $P(x)$ counted with multiplicities by formula (5). In a natural way $P(x)$ induces a map $\tilde{P}(x, w)$ between covering spaces in the diagram

$$\begin{array}{ccc} (x, w) \in M(\mathbf{e}) & \xrightarrow{\tilde{P}} & \mathbb{CP}^1 \ni u \\ \Big\downarrow \chi & & \Big\downarrow \sigma \\ x \in \mathbb{CP}^1 & \xrightarrow{P} & \mathbb{CP}^1 \end{array} \tag{8}$$

where $\chi(x, w) := x$ is a two–sheeted covering ramified over the points in $\mathbf{e}$ and $\sigma(u) := \frac{1}{2}(u + 1/u)$ is a two–sheeted covering ramified over ± 1. The map $\tilde{P}(x, w)$ intertwines deck transformations of the covering spaces: $\tilde{P}(x, -w) = 1/\tilde{P}(x, w)$, so the divisor of $\tilde{P}(x, w)$ consists of two points: a pole of order $n := \deg P$ at infinity ∞_+ on one sheet and a zero of the same order at infinity ∞_- on the other sheet. The converse result also holds: any (normalized) meromorphic function $\tilde{P}$ on $M(\mathbf{e})$ with divisor $n(\infty_- - \infty_+)$ satisfies the intertwining condition and induces a map $P(x)$ between the bases. This latter is a polynomial of degree n which a fortiori has the required number of simple critical points corresponding to the values ± 1.

In the framework of this construction the problem of describing g–extremal polynomials of fixed degree n is equivalent to describing all the curves M of the form (7) possessing a mermorphic function with divisor $n(\infty_- - \infty_+)$. The problem

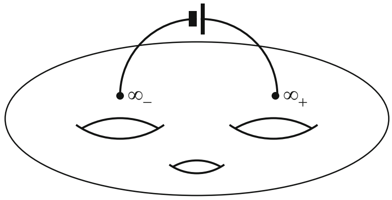

Fig. 1 An abelian integral of the 3rd kind on a Riemann surface is a complex potential of a flow with one source and one sink

of existence and representation for such a function is solved in terms of the curve M itself, with the help of Abel's criterion [69]. On each curve (7) there exists a unique abelian differential of the third kind

$$\eta_M = \left(x^g + \sum_{s=0}^{g-1} c_s x^s \right) \frac{dx}{w} \tag{9}$$

with purely imaginary periods. This normalization has a physical interpretation due to Helmholz: suppose that the surface is made of conducting material and attach infinitely thin wires to the points $\infty_\pm$ on the surface. Once the wires are connected to the battery, electric current arises along the surface, whose potential is the real part of the corresponding abelian integral $\int \eta_M$ (Fig. 1).

From the distinguished 1–form η_M on M we can recover the polynomial $P(x)$ of degree n up to a sign, by the explicit formula

$$P(x) = \pm \cos\left(ni \int_{(e,0)}^{(x,w)} \eta_M \right), \quad x \in \mathbb{C}, (x, w) \in M. \tag{10}$$

The result of the calculation of the right–hand side of (10) is independent of the choice of the path of integration, the branch point e, and the point $(x, w) \in M$ lying over x, the argument of the polynomial. This formula is a generalization of the classical representations (3), (4) for Chebyshev and Zolotarëv polynomials and Peherstorfer's representation for (non-classical) Chebyshev polynomials on several intervals [116]. It describes g–extremal polynomials using just a few parameters, the moduli of the curve M. However, these moduli are not arbitrary; they must satisfy several relations.

For a curve M associated with a polynomial of degree n the differential form η_M coincides with $n^{-1} d \log \tilde{P}(x, w)$, so a curve M is associated with a polynomial of degree n if and only if the periods of the distinguished 1–form η_M on the curve lie in the lattice $2\pi i n^{-1}\mathbb{Z}$. For a real curve M, on which we can define the reflection $\bar{J}(x, w) := (\bar{x}, \bar{w})$, half of these periods must be zero, and the other half must satisfy the system of Abel's equations

$$-i \int_{C_s^-} \eta_M = 2\pi \frac{m_s}{n}, \quad s = 0, 1, \ldots, g, \tag{11}$$

where the m_s are integers and $\{C_s^-\}_{s=0}^g$ is a basis of the lattice of integral 1–cycles on M changing sign after the reflection $\bar{J}$.

Chapter 3. It is convenient to assume that the curve M associated with a polynomial is a point in the moduli space of real hyperelliptic curves of genus g with marked point ∞_+ on an oriented real oval. This space consists of several components $\mathscr{H}_g^k$, $k = 0, \ldots, g+1$, distinguished by the number of real points in the variable branch divisor **e**. Each component of the moduli space is a smooth 2g–dimensional real manifold homeomorphic to the product of a cell and the *configuration space* of the (half)plane. The fundamental group of $\mathscr{H}_g^k$, which is isomorphic to the Artin *braid group* on $g-k+1$ strands, acts on the universal cover $\tilde{\mathscr{H}}_g^k$ of the moduli space by deck transformations. We consider four representations (models) for $\tilde{\mathscr{H}}_g^k \cong \mathbb{R}^{2g}$; one of these, an analytic uniformization of the moduli space $\mathscr{H}_g^k$, is used in what follows for the effective calculation of extremal polynomials.

Chapter 4. To visualize the description of all extremal polynomials we develop graph technique. A point M in the moduli space defines the *horizontal foliation* of the distinguished quadratic differential $(\eta_M)^2$ on the Riemann sphere. Strebel's theory [143] is concerned with such foliations. The foliation in question is orthogonal to level curves of the function $W(x) := \left|\mathrm{Re}\int_{(e,0)}^{(x,w)} \eta_M\right|$, which is globally defined on the Riemann sphere. As a result, the structure of the horizontal foliation $(\eta_M)^2 > 0$ is rather simple: its trajectories do not form cycles or mix. The union of certain pieces of critical trajectories of the foliation and the zero level set of the function $W(x)$ forms a graph Γ; its edges are labelled by their lengths in the metric associated with the quadratic differential. All the restrictions on the topology and weights of the graph Γ are easy to write out, so an abstract weighted graph Γ satisfying these conditions can be realized as the graph associated with a unique curve M in the moduli space. Fixing the topology of the graphs Γ, but considering variable weights we decompose each component $\mathscr{H}_g^k$ of the moduli space into finitely many cells $\mathscr{A}[\Gamma]$, in which the periods of the distinguished abelian differential η_M on the curve form a part of a natural system of coordinates. In particular, in each cell $\mathscr{A}[\Gamma]$ the curves M in the moduli space which are associated with polynomials of fixed degree can be described by the system of linear equations in this natural system of coordinates (11).

Chapter 5. In a fixed moduli space $\mathscr{H}_g^k$ we investigate Abel's equations (11), which describe the points associated with polynomials of degree n. Integrals of the differential form η_M over independent odd 1–cycles $C^- := -\bar{J}C^-$ on M define locally the *period map* on $\mathscr{H}_g^k$. Usually, the moduli space is not simply connected and the period map cannot be extended to a global map because going about a non-trivial cycle in the moduli space results in a change of basis in the lattice of odd 1–cycles on the curve. The resulting monodromy is described by the Burau representation ([28], 1932) of the braid group; it vanishes after passing to the universal cover $\tilde{\mathscr{H}}_g^k$. At points in the universal cover we have a distinguished basis of the odd homology space: it can be obtained by the parallel translation of a fixed basis at the marked point M_0 which corresponds to a natural flat Gauss–Manin connection in the vector bundle of homology spaces. Then the left–hand sides of the equations in (11) define a global period map from the universal cover into the $(g+1)$–dimensional real Euclidean space. We shall explicitly find the image of the universal covering space $\tilde{\mathcal{H}}_g^k$ under this map: for $k = g+1$ this is the interior of a g–simplex, for $k = g$ a union of k open simplexes, and for $k < g$ an infinite countable union of open g–simplexes numbered by braids. We also show that the period map is a submersion. In particular, the points in the moduli space associated with polynomials of degree n form smooth manifolds $\mathbb{T}(\cdot)$ of dimension g, which correspond to the lattice on the right–hand sides of Abel's (11). These manifolds are dense in the moduli space in the limit as $n \to \infty$.

Chapter 6 is concerned with effective calculations in the moduli space for two problems:

(1) Solving Abel's equations (11).
(2) Recovering extremal polynomials by formula (10) and recovering their derivatives of various orders to satisfy the constraints of the least deviation problem.

To this end we uniformize the curves M, points in the moduli space, by certain special Schottky groups. Summing Poincaré theta series we obtain abelian differentials on curves and, in particular, η_M. Abel's equations and our representation for extremal polynomials can be re–written in terms of global coordinate variables on the universal cover $\tilde{\mathscr{H}}_g^k$ which are related to the parameters of the generators of the Schottky group.

Chapter 7. How can we solve any given least deviation problems using our approach? We propose to make use of a suitable Ansatz. *First* we must analyse the problem and find the discrete parameters of the Ansatz: the topological invariants g and k and the indices $m_0, m_1, \ldots, m_g$ corresponding to the low-dimensional face of a ball in the space of polynomials (2) that contains the solution. *Next* we must set and solve numerically a system of $2g$ transcendental equations for the point $M \in \mathscr{H}_g^k$ in the moduli space associated with the solution $P_n(x)$. Abel's equations describe a smooth submanifold of the moduli space which is (locally) parametrized by g coordinate functions, the continuous parameters of the Ansatz. These can be found with the help of the data of the extremal problem, the constraints on the coefficients of the polynomials and the end-points of the set E. The variational formulae in Chap. 6 allow us to arrange various versions of descent methods [31] to solve the problem of navigation in the moduli space.

We consider the above scheme to solve optimization problems in greater detail, taking for example the calculation of the optimal stability polynomial $R_n(x)$ approximating the exponent to order $p = 3$ at the origin and deviating from zero at most by 1 on the maximal possible interval on the real axis. Problem B for $p = 3$ and any degree n reduces to the solution of 4 equations in the 4–dimensional moduli space of algebraic curves $\mathscr{H}_2^1$. Keeping in mind the reader who is only interested in numerical applications we have written this chapter to be understandable to those who have not read the other chapters of the book.

The **Conclusion** contains a list of open problems.