Lecture Notes in Physics

Volume 845

For further volumes:
http://www.springer.com/series/5304

The Lecture Notes in Physics

The series Lecture Notes in Physics (LNP), founded in 1969, reports new developments in physics research and teaching—quickly and informally, but with a high quality and the explicit aim to summarize and communicate current knowledge in an accessible way. Books published in this series are conceived as bridging material between advanced graduate textbooks and the forefront of research and to serve three purposes:

- to be a compact and modern up-to-date source of reference on a well-defined topic
- to serve as an accessible introduction to the field to postgraduate students and nonspecialist researchers from related areas
- to be a source of advanced teaching material for specialized seminars, courses and schools

Both monographs and multi-author volumes will be considered for publication. Edited volumes should, however, consist of a very limited number of contributions only. Proceedings will not be considered for LNP.

Volumes published in LNP are disseminated both in print and in electronic formats, the electronic archive being available at springerlink.com. The series content is indexed, abstracted and referenced by many abstracting and information services, bibliographic networks, subscription agencies, library networks, and consortia.

Proposals should be sent to a member of the Editorial Board, or directly to the managing editor at Springer:

Christian Caron
Springer Heidelberg
Physics Editorial Department I
Tiergartenstrasse 17
69121 Heidelberg/Germany
christian.caron@springer.com

Mauro Carfora · Annalisa Marzuoli

Quantum Triangulations

Moduli Spaces, Strings, and Quantum Computing

Springer

Prof. Dr. Mauro Carfora
Dipto. Fisica Nucleare e Teorica
Istituto Nazionale di Fisica
Nucleare e Teorica, Sez. di Pavia
Università degli Studi di Pavia
Via Bassi 6
27100 Pavia
Italy

Prof. Annalisa Marzuoli
Dipto. Fisica Nucleare e Teorica
Istituto Nazionale di Fisica
Nucleare e Teorica, Sez. di Pavia
Università degli Studi di Pavia
Via Bassi 6
27100 Pavia
Italy

Present Address
Dipto. di Matematica "F. Casorati"
Università degli Studi di Pavia
Via Ferrata 1
27100 Pavia
Italy

ISSN 0075-8450
e-ISSN 1616-6361
ISBN 978-3-642-24439-1
e-ISBN 978-3-642-24440-7
DOI 10.1007/978-3-642-24440-7
Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2011941664

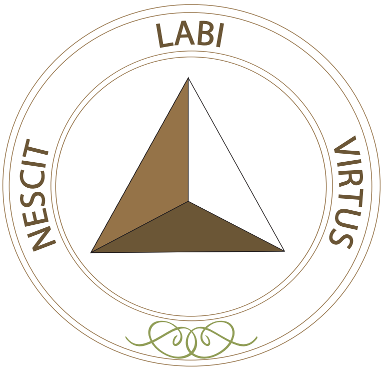
Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

...These are the causes of the formation of pure primary bodies. The presence in each kind of further varieties is due to the construction of the two basic triangles. This originally produces triangles not in one size only, but some smaller and some larger, the number of sizes corresponding to the number of varieties. So their combinations with themselves and with each other give rise to endless complexities, which anyone who is to give a likely account of reality must survey

Plato, Timaeus
(translated by Desmond Lee,
Penguin, London (1965))

Preface



The above illustration shows a variant woodcut printer's device on verso last leaf of rare XVI century edition of Plato's Timaeus, *(Divini Platonis Operum a Marsilio Ficino tralatorum, Tomus Quartus. Lugduni, apud Joan Tornaesium M.D.XXXXX)*. The printer's device to the colophon shows a medaillon with a tetrahedron in centre, and the motto round the border: *Nescit Labi Virtus,* Virtue Cannot Fail. This woodcut beautifully illustrates the role of the perfect shape of the tetrahedron in classical culture. The tetrahedron conveys such an impression of strong stability as to be considered as an epithome of virtue, unfailingly capturing us with the depth and elegance of its shape. In the course of history the geometry of the tetrahedron, of the Platonic solids and more generally of the highly symmetrical discrete patterns one encounters in Nature and Art has always been connected with some of the more sophisticated aspects of Mathematics and Physics of the time. From Plato's *Timaeus*, to Piero della Francesca's *Libellus De Quinque Corporibus Regularibus*, to Pacioli's *De Divina Proportione*, up to Kepler's *Harmonices Mundi* there have always been attempts to use the Platonic solids and their many variants to provide mathematical models of the physical universe. What makes these shapes perfectly irresistible to many mathematicians and physicists, both amateur and professional, is culturally related to their long–standing role in natural philosophy, but also to the deceptive fact that the geometry of these discrete structures often points to unexpected connections between very distinct aspects of Mathematics and Physics. A property, this latter, that modern theoretical physics has brought attention to even more. Indeed, polyhedral manifolds, the natural generalization of Platonic Solids, play quite a distinguished role in such settings as Riemann moduli space theory, strings and quantum gravity, topological quantum field theory, condensed matter physics, and critical phenomena. The motivation of such a wide spectrum of applications goes beyond the observation that polyhedral manifolds provide a natural discrete analogue of the smooth manifolds in which a physical theory is framed. Rather, it is often a consequence of an underlying structure, only apparently combinatorial, which naturally calls into play non-trivial aspects of representation theory, of complex analysis, and topology in a way which makes manifest the basic geometric structures of the physical interactions involved. In spite of these remarks, one has to admit that in almost all existing literature, the role of triangulated manifolds remains that of a convenient discretization of the physical theory, a grab-bag of techniques which are computationally rather than conceptually apt to disclose the underlying physics and geometry. The restriction to such a computational role may indeed be justified by the physical nature of the problem, as is often the case in critical statistical field theory, but sometimes it is not. This is the discriminating criterion motivating this Lecture Notes, since in the broad panorama the theory offers, the relation between polyhedral surfaces, Riemann moduli spaces, non-critical string theory, and quantum computing emerges as a clear path probing the connection between triangulated manifolds and quantum physics to the deepest.

Chapter 1 is devoted to a detailed study of the geometry of polyhedral manifolds, in particular of triangulated surfaces. This subject, which may be considered a classic, has recently seen a flourishing of many new results of great potential impact in the physical applications of the theory. Here the focus is on results which are either new or not readily accessible in the standard repertoire. In particular we discuss from an original perspective the structure of the space of all polyhedral surfaces of a given genus and their stable degenerations. In such a framework, and in the whole landscaping of the space of polyhedral surfaces, an important role is played by the conical singularities associated with the Euclidean triangulation of a surface. We provide a detailed analysis of the geometry of these singularities, introduce the associated notion of cotangent cones, circle bundles, and of the attendant Euler class on the space of polyhedral surfaces. This is a rather delicate point which appears in many guises in quantum gravity, and string theory, and which is related to the role that Riemann moduli space plays in these theories. Not surprisingly, the Witten–Kontsevich model lurks in the background of our analysis, and some of the notions we introduce may well serve for illustrating, from a more elementary point of view, the often deceptive and very technical definitions that characterize this subject.

We turn in Chap. 2 to the formulation of a powerful dictionary between polyhedral surfaces and complex geometry. It must be noted that, both in the mathematical and in the physical applications of the theory, the connection between Riemann surfaces and triangulations typically emphasizes the role of ribbon graphs and of the associated metric. The conical geometry of the polyhedral surface is left aside and seems to play no significant a role. This attitude can be motivated by Troyanov's basic observation that the conformal structure does not see the conical singularities of a polyhedral surface. However, this gives a narrow perspective of the much wider role that the theory has to offer. Thus, we connect a polyhedral surface to a corresponding Riemann surface by taking fully into account its conical geometry. This connection is many-faceted and exploits a vast repertoire of notion ranging from complex function theory to algebraic geometry. We start by defining the barycentrically dual polytope associated with a polyhedral surface and discuss the geometry of the corresponding ribbon graph. By adapting to our case an elegant version of Strebel theorem provided by Mulase, we explicitly construct the Riemann surface associated with the dual polytope. This directly bring us to the analysis of Troyanov's singular Euclidean structures and to the construction of the bijective map between the moduli space $\mathfrak{M}_{g,N_0}$ of Riemann surfaces (M, N_0) with N_0 marked points, decorated with conical angles, and the space of polyhedral structures. In particular the first Chern class of the line bundles naturally defined over $\mathfrak{M}_{g,N_0}$ by the cotangent space at the ith marked point is related with the corresponding Euler class of the circle bundles over the space of polyhedral surfaces defined by the conical cotangent spaces at the ith vertex of the triangulation. Whereas this is not an unexpected connection, the analogy with Witten–Kontsevich theory being obvious, we stress that the conical geometry adds to this property the possibility of a deep and explicit characterization of the Weil–Petersson form in terms of the edge-lengths of the triangulation. This result is obtained by a subtle interplay between the geometry of polyhedral surfaces and 3-dimensional hyperbolic geometry, and it will be discussed in detail in Chap. 3 since it explicitly hints to the connection between polyhedral surfaces and quantum geometry in higher dimensions.

As we said before, Chap. 3 deals with the interplay between polyhedral surfaces and 3-dimensional hyperbolic geometry, and to the characterization of the Weil–Petersson form ω_{WP} on the space of polyhedral structures with given conical singularities. An important role in such a setting is played by the recent nice results by G. Mondello on an explicit expression of the Weil–Petersson form for hyperbolic surfaces with geodesic boundaries. In order to construct a combinatorial representative of ω_{WP} for polyhedral surfaces we exploit this result and the connection between similarity classes of Euclidean triangles and the triangulations of 3-manifolds by ideal tetrahedra. We describe this construction in detail since it will also characterize a striking mapping between closed polyhedral surfaces and hyperbolic surfaces with geodesic boundaries. Such a mapping has a life of its own strongly related with the geometry of moduli space of pointed Riemann surfaces and it provides a useful framework for discussing such matter as open/closed string dualities.

The content of Chap. 4 constitutes an introduction to the basic ideas of two-dimensional quantum field theory and non-critical strings. This is a classic which however is useful for illustrating the interplay between Quantum Field Theory, moduli space of Riemann surfaces, and the properties of polyhedral surfaces which is the *leitmotiv* of this LNP. At the root of this interplay there is 2D quantum gravity. It is well know that such a theory allows for two complementary descriptions: On one side we have a conformal field theory (CFT) living on a 2D world-sheet, a description that emphasizes the geometrical aspects of the Riemann surface associated with the world-sheet; on the other side, the theory can be formulated as a statistical critical field theory over the space of polyhedral surface (dynamical triangulations). We show that many properties of such 2D quantum gravity models are connected with a geometrical mechanism which allows to describe a polyhedral surface with N_0 vertices as a Riemann surface with N_0 punctures dressed with a field whose charges describe discretized curvatures (connected with the deficit angles of the triangulation). Such a picture calls into play the (compactified) moduli space of genus g Riemann surfaces with N_0 punctures $\mathfrak{M}_{g;N_0}$ and allows to prove that the partition function of 2D quantum gravity is directly related to the computation of the Weil–Petersson volume of $\mathfrak{M}_{g;N_0}$. By exploiting the large N_0 asymptotics of the such Weil–Petersson volume, recently characterized by Manin and Zograf, it is then easy to connect the anomalous scaling properties of pure 2D quantum gravity, the KPZ exponent, to the Weil–Petersson volume of $\mathfrak{M}_{g;N_0}$. This ultimately relates with the difficult problem of constructively characterizing the appropriate functional measures on spaces of Riemannian manifolds often needed in the study of quantum gravity models and in the statistical mechanics of extended objects. We also address the more general case of the interaction of conformal matter with 2D quantum gravity, and in particular the characterization of the associated KPZ exponents. By elaborating on the recent remarkable approach by A. Kokotov to the spectral theory over polyhedral surfaces we provide a general framework for analyzing KPZ exponents by discussing the scaling properties of the corresponding discretized Liouville theory.

In a rather general sense, polyhedral surfaces provide also a natural kinematical framework within which we can discuss open/closed string duality. A basic problem in such a setting is to provide an explanation of how open/closed duality is dynamically generated. In particular how a closed surface is related to a corresponding open surface, with gauge-decorated boundaries, in such a way that the quantization of such a correspondence leads to a open/closed duality. Typically, the natural candidate for such a mapping is Strebel's theorem which allows to reconstruct a closed N-pointed Riemann surfaces M of genus g out of the datum of a the quadratic differential associated with a ribbon graph. Are ribbon graphs, with the attendant BCFT techniques, the only key for addressing the combinatorial aspects of Open/Closed String Duality? The results of Chap. 3 show that from a closed polyhedral surface we naturally get an open hyperbolic surface with geodesic boundaries. This gives a geometrical mechanism describing the transition between closed and open surfaces which, in a dynamical sense, is more interesting than Strebel's construction. Such a correspondence between closed polyhedral surfaces and open hyperbolic surface is indeed easily promoted to the corresponding moduli spaces: $\mathfrak{M}_{g;\,N_0} \times \mathbb{R}^N_+$ the moduli spaces of N_0-pointed closed Riemann surfaces of genus g whose marked points are decorated with the given set of conical angles, and $\mathfrak{M}_{g;\,N_0}(L) \times \mathbb{R}^{N_0}_+$ the moduli spaces of open Riemann surfaces of genus g with N_0 geodesic boundaries decorated by the corresponding lengths. Such a correspondence provides a nice kinematical set up for establishing a open/closed string duality, by exploiting the recent striking results by M. Mirzakhani on the relation between intersection theory over $\mathfrak{M}(g; N_0)$ and the geometry of hyperbolic surfaces with geodesic boundaries. The results in this chapter directly connect with many deep issues in 3-D geometry ultimately relating with the volume conjecture in hyperbolic geometry and with the role of knots invariants. This eventually bring us to the next topic we discuss.

Indeed, Chaps. 5 and 6 deal with the interplay between triangulated manifolds, Knots, Topological Quantum Field Theory, and Quantum Computation. As Justin Roberts has nicely emphasized, the standard topological invariants were created in order to distinguish between things and, owing to their intrinsic definitions, it is clear what kind of properties they reflect. For instance, the Euler number χ of a smooth, closed and oriented surface $\mathscr{S}$ determines completely its topological type and can be defined as $\chi\,(\mathscr{S}) = 2 - 2g$, where g is the number of handles of $\mathscr{S}$. On the other hand, quantum invariants of knots and 3-manifolds were *discovered*, but their indirect construction based on quantum group technology often hides information about the purely topological properties they are able to detect. What is lost at the topological level is however well paid back by the possibility of bridging this theory with a plenty of issues in pure mathematics and theoretical physics. To the early connections such as quantum inverse scattering and exact solvable models it is worth adding the operator algebra approach used originally by Jones in defining his knot polynomial. However, the most profitable development of the theory was that suggested by Schwarz and formalized by Witten. Indeed, recognizing quantum invariants as partition functions and vacuum expectation values of physical observables in Chern–Simons–Witten topological quantum field theory provides a *physical* explanation of their existence and properties. Even more radically, one could speak of a conceptual explanation, as far as the topological origin of these invariants keeps on being unknown. In this wider sense, quantum topology might be thought of as the mathematical substratum of an $SU(2)$ CSW topological field theory quantized according to the path integral prescription (the coupling constant $k \geq 1$ is constrained to be an integer related to the deformation parameter q by $q = \exp\left(\frac{2\pi i}{k+2}\right)$).

The CSW environment provides not only the physical interpretation of quantum invariants but it does include as well all the historically distinct definitions. In particular, monodromy representations of the braid group appear in a variety of conformal field theories since point-like ‘particles’ confined in 2-dimensional regions evolve along braided worldlines. As a matter of fact, the natural extension of CSW theory to a 3-manifold $\mathscr{M}^3$ endowed with a non empty 2-dimensional boundary $\partial\mathscr{M}^3$ induces on $\partial\mathscr{M}^3$ a specific quantized boundary conformal field theory, namely the $SU(2)$ Wess–Zumino–Witten (WZW) theory at level $\ell = k + 2$. The latter provides in turn the framework for dealing with $SU(2)_q$–colored links presented as closures of oriented braids and associated with Kaul unitary representation of the braid group. A further extension of this representation proposed can be used to construct explicitly the quantum 3-manifold invariants within a purely algebraic setting. Such quantities are essentially the Reshetikhin–Turaev–Witten invariants evaluated for 3-manifolds presented as complements of knots/links in the 3-sphere S^3, up to an overall normalization. Discretizations of manifolds appear here at a fundamental level, in particular from $SU(2)$-decorated triangulations of 3-dimensional manifolds to triangulated boundary surfaces supporting a (boundary) Conformal Field Theory. Their use is relevant both in the characterization of the theory and in the actual possibility of computing the topological invariants under discussion. This computational role is a basic property since the possibility of computing quantities of topological or geometric nature was recognized as a major achievement for quantum information theory by the Fields medalist Michael Freedman and co-workers. Their *topological quantum computation* setting was designed to comply with the behavior of *modular functors* of 3D Chern–Simons–Witten (CSW) non-abelian topological quantum field theory (TQFT) the gauge group being typically $SU(2)$. In physicists' language, such functors are partition functions and correlators of the quantum theory and, owing to gauge invariance and invariance under diffeomorphisms, which freeze out local degrees of freedom, they share a global, topological character. More precisely, the physical observables are associated with topological invariants of knots—the prototype of which is the Jones polynomial—and the generating functional is an invariant of the 3-dimensional ambient manifold, the Reshetikhin–Turaev–Witten invariant. We will discuss these matters in detail, with many illustrative examples and diagrams.

We think that these case studies illustrate well the richness of the subject with a repertoire of mathematical techniques and physical concepts that may disclose new exciting territories of research.

Pavia, April 2011

Mauro Carfora
Annalisa Marzuoli

Acknowledgments

We would like to thank Gaia for her patience with us over the years, our friends and colleagues Jan Ambjørn, Enzo Aquilanti, Giorgio Immirzi, Robert Littlejohn, Mario Rasetti, Tullio Regge, and our young collaborators for so many discussions about the polyhedral side of Mathematics and Physics.

Contents