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# Factors and Factorizations of Graphs

## Proof Techniques in Factor Theory

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to Frank Harary, my teacher
and
to my wife Yoko

# Preface

A spanning subgraph of a graph is its subgraph whose vertex set is the same as the original graph. In this book, spanning subgraphs of graphs possessing some given properties are studied, and these spanning subgraphs are called *factors*. For example, for a positive integer $k$, a $k$-factor of a graph is its spanning subgraph each of whose vertices has a constant degree $k$. So a 1-factor is nothing but a perfect matching, where a matching in a graph $G$ is a subgraph of $G$ whose edges are pairwise disjoint and a perfect matching is a matching that covers all the vertices of $G$. Furthermore a 2-factor is a set of vertex-disjoint cycles which together cover the vertices of the graph. Similarly, a graph each of whose vertices has a constant degree $k$ is called a $k$-regular graph, and if $k$ is even, such a graph is said to be even regular. If the edges of a graph $G$ can be decomposed into $k$-factors, then $G$ is said to be $k$-factorable.

Petersen's results on graph factors and factorizations date back to the 19th century. Petersen's Theorem, which he proved in order to solve a problem on Diophantine equations, states: *Every even regular graph is 2-factorable.* Petersen then went on to prove another theorem: *Every 2-edge connected 3-regular graph has a 1-factor.* This theorem arose from a counterexample Petersen constructed to Tait's 'theorem': *Every 3-regular graph which has no bridge is 1-factorable.* This counterexample is the well-known Petersen graph. Petersen's theorems are byproducts of attempts to address problems outside of graph theory. Perhaps Petersen himself had no inkling that these theorems would open the door to a very promising area of graph theory.

Later came Hall's Marriage Theorem, a result obtained when Hall was studying the structure of subsets. *Let $G$ be a bipartite graph with bipartition $(A, B)$. Then $G$ has a matching that saturates $A$ (i.e., a matching covering all the vertices of $A$) if and only if $|N_G(S)| \geq |S|$ for all $S \subseteq A$.* König's Theorem followed: *Every regular bipartite graph is 1-factorable.* And then Tutte's 1-Factor Theorem: *A graph $G$ has a 1-factor if and only if $odd(G - S) \leq |S|$ for all $S \subset V(G)$.* These five theorems form the foundation of the study of factors and factorizations.

König's theorem was the result of a conscious effort to answer a graph theory problem: *Does every bipartite regular graph have a 1-factor?* It seems that, at that time, König already had a good idea of how graph factors could be applied. In his paper "On graphs and their applications in determinant theory and set theory", he starts by saying "The following article deals with problems from analysis situs, the theory of determinants and set theory." He then goes on to assert that in fact, the notion of a graph and its usefulness as a method of representation actually links these three disparate areas.

A second stage of development is marked by Tutte's $f$-Factor Theorem: *Let $G$ be a graph and $f : V(G) \to \mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$. Then $G$ has an $f$-factor (i.e., a spanning subgraph $F$ satisfying* $\deg_F(v) = f(v)$ *for all vertices $v$ of $G$) if and only if for all disjoint subsets $S$ and $T$ of $V(G)$,*

$$\sum_{x \in S} f(x) + \sum_{x \in T} \big(\deg_{G-S}(x) - f(x)\big) - q(S,T) \geq 0,$$

*where $q(S,T)$ denotes the number of components $C$ of $G-(S \cup T)$ such that* $\sum_{x \in V(C)} f(x) + e_G(C,T) \equiv 1 \pmod 2$ ; and Lovász's $(g, f)$-Factor Theorem: *Let $G$ be a graph and $g, f : V(G) \to \mathbb{Z}$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then $G$ has a $(g,f)$-factor (i.e., a spanning subgraph $H$ satisfying $g(v) \leq \deg_H(v) \leq f(v)$ for all vertices $v$ of $G$) if and only if for all disjoint subsets $S$ and $T$ of $V(G)$,*

$$\sum_{x \in S} f(x) + \sum_{x \in T} (\deg_{G-S}(x) - g(x)) - q^*(S,T) \geq 0,$$

*where $q^*(S,T)$ denotes the number of components $C$ of $G-(S \cup T)$ such that $g(x) = f(x)$ for all $x \in V(C)$ and* $\sum_{x \in V(C)} f(x) + e_G(C,T) \equiv 1 \pmod 2$. Many subsequent theorems were proved based on these results.

A third stage of development began in the 1980's and is marked by the introduction of the notions of semi-regular factors and component factors. Since that time, many substantial results have been obtained and much progress has been made in this area. The results are detailed in this book.

As far as we know, there was no comprehensive text on factors and factorizations when we started to write this book. This is one compelling reason for writing this volume. Since we wrote our survey paper entitled "Factors and Factorizations of Graphs" published in *Journal of Graph Theory*, vol. 9 (1985), we collected and analyzed most of the results in the area. In fact, we started to write this book ten years ago. A first version of this text coincided with KyotoCGGT2007.

This book also chronicles the development of mathematical graph theory in Japan, a development which began with many important results in factors and factorizations of graphs.

One of the main themes of this text is the observation that many theorems can be proved using only a few standard proof techniques. Namely,

(i) We reduce a given factor problem of a graph $G$ into a simpler factor problem of a related graph $G^*$, and apply a known theorem on a simpler factor to $G^*$.
(ii) We consider several standard cases; for example, when we consider regular factors, a standard case analysis is useful.
(iii) If a criterion for a graph to have a factor is given as "$f(X) \leq 0$ for all $X \subset V(G)$", then we consider the following two cases: (a) there exists a nonempty vertex set $S$ such that $f(S) = 0$; and (b) $f(X) < 0$ for all nonempty vertex sets $X$.
(iv) If a criterion for a graph to have a factor is given as "$\rho(G - X) \leq f(X)$ for all $X \subset V(G)$", then we define a number

$$\beta = \min\{f(X) - \rho(G - X) \, : \, \emptyset \neq X \subset V(G), \; \rho(G - X) > 0\}.$$

Next, we consider a maximal vertex set $S$ of $G$ such that $\rho(G - S) > 0$ and $f(S) - \rho(G - S) = \beta$.
(v) We use alternating paths or trails. These arguments were often used in early stages of the developments of the theory of graph factors; they were eventually replaced by the above methods, since the proofs using alternating paths or trails are usually long and complicated. However, this method is still useful for algorithms for finding the desired factors.

Other key features of this book are the following:

1. It is comprehensive and covers most of the important results since 1980.
2. It is self-contained. One who wants to begin research in graph factors and factorizations can confine himself to this one book to follow the history and development of the area, and to find conjectures and open problems.
3. Many detailed illustrations are given to accompany the proofs.

As we mentioned above, the first version of this book was privately published on the occasion of KyotoCGGT2007 conference in honor of Jin Akiyama and Vasěk Chvátal on their 60th birthdays. Many parts of the book have since been revised and two more chapters, "Component Factors" and "Spanning Trees", have been added.

Frank Harary predicted that graph theory would grow so much that each chapter of his book *Graph Theory* would eventually expand to become a book on its own. He was right. This book is an expansion of his *Chapter 9, Factorization.*

We also predict that the area of factors and factorizations will continue to grow because of many applications to BIBD, Steiner Designs, Matching Theory, OR, etc. that have been found.

The following references were very helpful to us during the preparation of this work:

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