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Fourier Analysis and Nonlinear Partial Differential Equations

 Springer

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A la mémoire de Noomann Bassou

Preface

Since the 1980s, Fourier analysis methods have become of ever greater interest in the study of linear and nonlinear partial differential equations. In particular, techniques based on *Littlewood–Paley decomposition* have proven to be very efficient in the study of evolution equations. Littlewood–Paley decomposition originates with Littlewood and Paley’s works in the early 1930s and provides an elementary device for splitting a (possibly rough) function into a sequence of spectrally well localized smooth functions. In particular, differentiation acts almost as a multiplication on each term of the sequence. However, its systematic use for nonlinear partial differential equations is rather recent. In this context, the main breakthrough was achieved after J.-M. Bony introduced the *paradifferential calculus* in his pioneering 1981 paper (see [39]) and its avatar, the *paraproduct*.

Surprisingly, despite the growing number of authors who now use such techniques, to the best of our knowledge, there is no textbook presenting Fourier analysis tools in such a way that they may be directly used for solving nonlinear partial differential equations.

The aim of this book is threefold. First, we want to give a detailed presentation of harmonic analysis tools that are of constant use for solving nonlinear partial differential equations. Second, we want to convince the reader that the rough frequency splitting supplied by Littlewood–Paley decomposition (which turns out to be much simpler than, e.g., Calderon–Zygmund decomposition or wavelet theory) may still provide elementary and elegant proofs of some classical inequalities (such as Sobolev embedding and Gagliardo–Nirenberg or Hardy inequalities). Third, we give a few examples of how to use these basic Fourier analysis tools to solve linear or nonlinear evolution partial differential equations. We have chosen to present the most popular evolution equations, namely, transport and heat equations, (linear or quasilinear) symmetric hyperbolic systems, (linear, semilinear, or quasilinear) wave equations, and the (linear or semilinear) Schrödinger equation. We place a special emphasis on models coming from fluid mechanics (in particular, on the incompressible Navier–Stokes and Euler equations) for which, historically, the Littlewood–

Paley decomposition was first used. It goes without saying that our methods are also relevant for solving a variety of other equations. In fact, there has been a plethora of recent papers dedicated to more complicated nonlinear partial differential equations in which Littlewood–Paley decomposition proves to be a crucial tool.

This book is almost self-contained, inasmuch as having an undergraduate level understanding of analysis is the only prerequisite. There are rare exceptions where we have had to admit nontrivial mathematical results, in which case references are given. Apart from these, we have postponed references, historical background, and discussion of possible future developments to the end of each chapter. The book does not contain any definitively new results. However, we have tried to provide an exhaustivity that cannot be found in any single paper. Also, we have provided new proofs for some well-known results.

We have also decided not to discuss the theory of wavelets, even though this would be the natural extension of Littlewood–Paley decomposition. Indeed, it turns out that, to the best of our knowledge, there are almost no theoretical results for nonlinear partial differential equations in which wavelets cannot be replaced by a simple Littlewood–Paley decomposition.

When writing this book, we tried as much as possible to make a distinction between what may be proven by means of classical analysis tools and what really does require Littlewood–Paley decomposition (and the paraproduct). In fact, with only a few exceptions, all the material concerning Littlewood–Paley decomposition is contained in Chapter 2 so that the reader who is not accustomed to (or who is afraid of) those techniques may still read a great deal of the book. In fact, the whole of Chapter 1, the first section of Chapter 3, the first half of Chapter 4, Chapter 5 (except for the last section), the first section of Chapter 6, and the first two sections of Chapter 8 may be read completely independently of Chapter 2. In most of the other parts of the book, Chapter 2 may be used freely as a “black box” that does not need to be opened.

Roughly speaking, the book may be divided into two principal parts: Tools are developed in the first two chapters, then applied to a variety of linear and nonlinear partial differential equations (Chapters 3–10). A detailed plan of the book is as follows.

Chapter 1 is devoted to a self-contained elementary presentation of classical Fourier analysis results. Even though none of the results are new, some of the proofs that we present are not the standard ones and are likely to be useful in other contexts. We also pay attention to the construction of explicit examples which illustrate the optimality of some refined estimates.

In Chapter 2 we give a detailed presentation on Littlewood–Paley decomposition and define homogeneous and nonhomogeneous Besov spaces. We should emphasize that we have replaced the usual definition of homogeneous spaces (which are quotient distribution spaces modulo polynomials) by something better adapted to the study of partial differential equations (indeed, dealing with distributions modulo polynomials is not appropriate in this con-

text). We also establish technical results (commutator estimates and functional inequalities, in particular) which will be used in the following chapters.

In Chapter 3 we give a very complete theory of strong solutions for transport and transport-diffusion equations. In particular, we provide a priori estimates which are the key to solving nonlinear systems coming from fluid mechanics. Chapter 4 is devoted to solving linear and quasilinear symmetric systems with data in Sobolev spaces. Blow-up criteria and results concerning the continuity of the flow map are also given. The case of data with critical regularity (in a Besov space) is also investigated.

In Chapter 5 we take advantage of the tools introduced in the previous chapters to establish most of the classical results concerning the well-posedness of the incompressible Navier–Stokes system for data with critical regularity. In order to emphasize the robustness of the tools that have been introduced hitherto in this book, we present in Chapter 6 a nonlinear system of partial differential equations with degenerate parabolicity. In fact, we show that some of the classical results for the Navier–Stokes system may be extended to the case where there is no vertical diffusion. Most of the results of this chapter are based on the use of an *anisotropic* Littlewood–Paley decomposition.

Chapter 7 is the natural continuation of the previous chapter: The diffusion term is removed, leading to the study of the Euler system for inviscid incompressible fluids. Here, we state local (in dimension $d \geq 3$) and global (in dimension two) well-posedness results for data in general Besov spaces. In particular, we study the case where the data belong to Besov spaces for which the embedding in the set of Lipschitz functions is critical. In the two-dimensional case, we also give results concerning the inviscid limit. We stress the case of data with (generalized) vortex patch structure.

Chapter 8 is devoted to *Strichartz estimates* for dispersive equations with a focus on Schrödinger and wave equations. After proving a dispersive inequality (i.e., decay in time of the L^∞ norm in space) for these equations, we present, in a self-contained way, the celebrated TT^* argument based on a duality method and on bilinear estimates. Some examples of applications to semilinear Schrödinger and wave equations are given at the end of the chapter.

Chapter 9 is devoted to the study of a class of quasilinear wave equations which can be seen as a toy model for the Einstein equations. First, by taking advantage of energy methods in the spirit of those of Chapter 4, we establish local well-posedness for “smooth” initial data (i.e., for data in Sobolev spaces embedded in the set of Lipschitz functions). Next, we weaken our regularity assumptions by taking advantage of the dispersive nature of the wave equation. The key to that improvement is a quasilinear Strichartz estimate and a refinement of the paradifferential calculus. To prove the quasilinear Strichartz estimate, we use a microlocal decomposition of the time interval (i.e., a decomposition in some interval, the length of which depends on the size of the frequency) and geometrical optics.

In Chapter 10 we present a more complicated system of partial differential equations coming from fluid mechanics, the so-called *barotropic compressible*

Navier–Stokes equations. Those equations are of mixed hyperbolic-parabolic type. We show how we may take advantage of the results of Chapter 3 and the techniques introduced in Chapter 2 so as to obtain local (or global) unique solutions with critical regularity. The last part of this chapter is dedicated to the study of the *low Mach number limit* for this system. It is shown that under appropriate assumptions on the data, the limit solution satisfies the incompressible Navier–Stokes system studied in Chapter 5.

In writing this book, we had help from many colleagues. We are particularly indebted to F. Charve, B. Ducomet, C. Fermanian-Kammerer, F. Sueur, B. Texier, and to the anonymous referees for pointing out numerous mistakes and giving suggestions and advice. In addition to J.-M. Bony, our work was inspired by many collaborators and great mathematicians, among them B. Desjardins, I. Gallagher, P. Gérard, E. Grenier, T. Hmidi, D. Iftimie, H. Koch, S. Klainerman, Y. Meyer, M. Paicu, D. Tataru, F. Vigneron, C.J. Xu, and P. Zhang. We would like to express our gratitude to all of them.

Paris

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