Part IV

Stochastic Differential Equations

The theory of SDEs has raised the probability space of Brownian motion to the status of a *universal probability space* on which the semigroup generated by any elliptic operator can be realized; this realization can be done in a canonical form for elliptic operators which are written as a sum of squares of Lie derivatives along *driving vector fields*.

In the Stratonovich formalism, the theory of stochastic differential equations behaves under changes of variables like the theory of ordinary differential equations. We will go further than this formal analogy by enforcing the theory of SDE as a limiting case of the theory of ODE: this is the substance of the *limit theorem*.

The approximating ODEs are constructed from the *control map* which is generated by the driving vector fields. The structure of the control map depends heavily upon the Lie algebra $\mathcal{A}$ generated by the driving vector fields: when $\mathcal{A}$ is commutative, the solution of the SDE is locally equal to a Brownian motion; the non-commutativity of $\mathcal{A}$ induces a *hypoellipticity* phenomenon which constitutes one of the main difficulties in proving the limit theorem.

A by-product of the limit theorem is a *principle of transfer* which PROVES automatically results for SDE from classical results on ODE.

This transfer principle makes it possible to differentiate the solution of an SDE relative to its initial value. This *stochastic calculus of variations* has the consequence that solutions of SDEs are functionals on the Wiener space X which belong to $D_\infty(X)$: the results of Chapter III on the regularity of laws become applicable to fundamental solutions of parabolic equations. So Stochastic Analysis provides an alternative approach to what is known in classical PDE theory as parabolic regularity; this approach goes further in degenerate cases or in infinite dimension.