Part II

Quasi-Sure Analysis

In finite dimensions, a function belonging locally to all the Sobolev spaces is a C^∞ function. This fact is a consequence of the Bernstein-Sobolev embedding $H^s(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ for $s > d/2$. By contrast, $D_\infty(X)$ is not contained in any space of continuous functions. The quasi-sure analysis is built as a substitute on a Gaussian probability space for the continuity of the functions of D_∞.

A Borel measure on X is of *finite energy* if it defines a continuous linear form on $D_\infty(X)$. A Borel set is said to be *slim* if it does not carry a measure of finite energy. The basic σ-field of quasi-sure analysis will be the Borel σ-field "quotiented" by the slim sets. In classical probability, the basic σ-field is the Borel σ-field "quotiented" by the events of probability zero. As every slim set is of probability zero, quasi-sure analysis will be a refinement of classical probability theory.

The notion of continuity in finite dimensions will be replaced by the notion of *quasi-continuity*, which means continuity outside "almost negligible" sets. The notion of continuity will be formulated relative to the topology of a fixed numerical model. A basic issue is to prove that this definition is independent of the numerical model which has been fixed. The proof of this fact will result from the development in infinite dimensions of a non-linear potential theory. Quasi-sure analysis can be constructed on any abstract Wiener space; again we shall show the independence of this construction from the choice of a Gross radonifying norm.

Quasi-sure analysis makes it possible to build a *co-area formula* describing the disintegration of the Gaussian measure along an $\mathbb{R}^d$-valued, non-degenerate map. In quasi-sure analysis, the "leaves" of such a map exist for all $\xi \in \mathbb{R}^d$. The desired disintegration formula is based on a construction of the *area* for the *finite-codimensional submanifolds* constituted by the leaves. The computations of differential geometry on submanifolds of $\mathbb{R}^d$ can be transferred to finite-codimensional submanifolds of a Gaussian space.

It seems that an exposition of quasi-sure analysis appears here for the first time in book form. In the remainder of this book quasi-sure analysis is used only slightly: **As this Part II is highly technical, it can be skipped on first reading.**