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A Series of Comprehensive Studies in Mathematics

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Preface

An appropriate coverage of the subjects contained in the five parts of this book would need several monographs. We hope that the global treatment presented here may emphasize some of their deep interactions. As far as possible we present self-contained proofs; we have also tried to produce a book that could be used in a graduate course.

Our thread of Ariadne is the introduction into stochastic analysis of the methodology used in classical analysis and differential geometry. Our geometric point of view has obliged us to pay great attention to the foundations. On the other hand our notation, which follows the usual conventions, will allow an experienced worker to look directly at any section of this book, without spending time on the foundational sections.

Each part is constructed according to the following format: a short introduction, a detailed table of contents at the beginning of each chapter of that part and a short note on the literature at the end of each part.

The style of writing oscillates from one part to the next between that of a rather technical monograph in Part II to a broader survey style in Parts IV and V.

I owe a great debt to K. Itô, J.R. Norris, D.W. Stroock for their careful reading of the first draft and for their far-reaching suggestions.

Paris, January 1997

Paul Malliavin

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Contents by Chapter

Chapter I. Gaussian Probability Spaces

Definition of a Gaussian probability space, reducibility – Hermite polynomials on \mathbb{R} – Hermite polynomials on \mathbb{R}^N – Numerical model of a Gaussian probability space – Intrinsic geometry on a Gaussian probability space – The Ornstein-Uhlenbeck semigroup, chaos decomposition – The Cameron-Martin representation – Abstract Wiener space.

Chapter II. Gross-Stroock Sobolev Spaces over a Gaussian Probability Space

Continuity of the Cameron-Martin representation over $L^{\infty-0}$ – The space \mathbb{D}_1^∞ of differentiable vectors of the Cameron-Martin representation – Gradient operator – Generalized polynomials – Cauchy operator; Krée-Meyer inequality for the gradient – The spaces \mathbb{D}_1^p – Gradient of Hilbert-valued functionals – Recursive approach to higher derivatives and to Krée-Meyer inequalities of higher order – The space \mathbb{D}_∞ of smooth functionals, its approximation by C^∞ cylindrical functionals – Divergence as the adjoint of the gradient – Divergence of smooth vector fields – Shigekawa acyclicity of the complex of differential forms – Appendix: Proof of the L^p inequality for the Hilbert transform.

Chapter III. Smoothness of Laws

Divergence of differentiable flow – Divergence as the adjoint operator of a derivation – Non-degenerate maps and their covariance matrices – Lifting up vector fields through a non-degenerate map – Pushing down divergences – Hölder regularity under $\mathbb{D}_2^{\infty-0}$ hypothesis – Smoothness under \mathbb{D}_∞ hypothesis – Lifting up and pushing down through a non-degenerate \mathbb{D}_∞ map – Inverse image of a distribution – Absolute continuity of scalar functionals under \mathbb{D}_1^1 hypothesis – Appendix: Computation of derivatives by means of divergence operators – Law of a weakly non degenerated map.

Chapter IV. Foundations of Quasi-Sure Analysis: Hierarchy of Capacities and Precise Gaussian Probability Spaces

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tionals, their representation by a Borel measure – Equilibrium potentials – Continuity of capacities – Capacitability of Borel sets – Equilibrium measures – Measures of finite energy – Charge and capacity – Slim sets are null sets for all measures of finite energy – Invariance of capacities under a change of numerical model, precise Gaussian probability space – Quasi-sure analysis on an abstract Wiener space.

Chapter V. Differential Geometry on a Precise Gaussian Probability Space

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Chapter VIII. From Ordinary Differential Equations to Stochastic Flow: The Transfer Principle

Stratonovich SDE – Intrinsic stochastic integral – Itô SDE, uniqueness of the Cauchy problem – The Stroock-Varadhan piecewise linear approximation – Algebraic analysis on the group of C^∞ -diffeomorphisms: the exponential map, the

adjoint action – Reduced variation of the Stroock-Varadhan approximation sequence of a smooth SDE, its a priori bound; Bootstrap and proof of the limit theorem – Cauchy problem for SDE with Lipschitz coefficients – Critique of our approach.

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The Ornstein-Uhlenbeck (OU) flow on a finite-dimensional Gaussian space – Lifting the OU flow to an abstract Wiener space, its axiomatic definition – Representation of the OU flow on the probability space of Brownian motion – Itô calculus of variations along the OU flow – Equilibrium processes and regularity of law.

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