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Triangulations

Structures for Algorithms and Applications

With 550 Figures and 14 Tables

Springer

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Preface

Triangulations appear in many different parts of mathematics and computer science since they are the natural way to decompose a region of space into smaller, easy-to-handle pieces. From volume computations and meshing to algebra and topology, there are many natural situations in which one has a fixed set of points that can be used as vertices for the triangulation. Typically one wants to find an optimal triangulation of those points or to explore the set of their all triangulations. The given points may represent the "sites" for a Delaunay triangulation computation, the test points for a surface reconstruction, or a set of monomials, represented as lattice points in $\mathbb{Z}^d$, in an algebraic-geometric meaning.

A central theme of this book is to use the rich geometric structure of the space of triangulations of a given set of points to solve computational problems (e.g., counting the number of triangulations or finding optimal triangulations with respect to various criteria), and for setting up connections to novel applications in algebra, computer science, combinatorics, and optimization. Thus at the heart of the book is a comprehensive treatment of the theory of regular subdivisions, secondary polytopes, flips, chambers, and their interactions. Again, we firmly believe that understanding the fundaments of geometry and combinatorics pays up for algorithms and applications.

The book is designed to serve as a textbook or for self-guided study. It was written with graduate students or advanced undergraduates as the target audience (in fact, several groups of students were kind enough to let us test the book with them). Beyond good knowledge of linear algebra, all that is required to use this book is maturity to read and understand proofs. With many examples and exercises, and with over five hundred illustrations, we aim to gently introduce beginners to the properties of the spaces of triangulations of "highly-structured" (e.g., cubes, cyclic polytopes, lattice polytopes, etc.) and "pathological" situations (e.g., disconnected spaces of triangulations, NP-hardness constructions, etc.). We do this in arbitrary dimension, while using only elementary geometric principles. We are excited to present many open questions. Some are new, but many have been open for some time. Also, the book contains many new results appearing here for the first time, besides corrections and simpler proofs of well-known theorems.

Chapter 1 describes several instances where triangulations of a point set naturally arise in combinatorics, optimization, and algebra, as motivation for the rest. A reader may select which parts he or she is most interested in and skip the rest. None of the material is a prerequisite for later chapters, but we hope to communicate some of the exciting and diverse applications possible and to show that triangulations are worth studying by outsiders.

Chapters 2 and 4 lay out the formal language, notation, and basic constructions. Concerning the language, we have decided to distinguish between the points (or vectors) of a configuration and the labels used to denote them. This may look awkward to the beginner at first sight but it has many advantages in the long run.

Chapter 3, is an "interlude" devoted specifically to what happens in two dimensions and a quick glance at dimension three. This chapter is almost independent from the rest and we hope it will help the reader to build intuition and to motivate, in a visual way, the challenges to come in arbitrary dimension (e.g., the notion of flip, enumeration, optimization, etc). Because Chapter 3 lies in between two more technical chapters we included more examples and applications that helped balance the presentation. It is a fun detour through a very active area of computational geometry.

Chapter 5 contains what is probably the central theorem of the book: Gelfand, Kapranov, and Zelevinsky's ground-breaking construction of a polytope with face lattice equal to the poset of *regular subdivisions* for any given configuration. This theorem is the central tool for flipping algorithms such as the incremental randomized construction of Delaunay triangulations, customary in Computational Geometry.

The next two chapters are devoted to the study of important examples of configurations (Chapter 6) and triangulations (Chapter 7). In the first one, nice combinatorial structures allow for a deeper study of the properties of these examples, while in the latter the focus is on ingenious constructions of pathological triangulations (disconnected flip-graphs, and triangulations with very few or no flips).

Chapter 8 focuses on computation and algorithms. We start with a discussion of data structures and discuss methods for enumeration and optimization in triangulations of arbitrary dimension. Here we prove that the structural understanding helps with the design algorithms, software, and the analysis of computational complexity.

Finally, Chapter 9 explores generalizations or different ways of looking at some of the structures in the rest of the book. Some of these "further topics" are so rich they could be themselves central topics of books. Interesting directions discussed include fiber polytopes, Cayley's construction, Gröbner bases, connections to lattice points and Ehrhart functions, and the combinatorics of simplicial spheres.

If you are a teacher planning to give a one semester course based on this book, the core of it should be Chapters 2, 4 and 5, ending with Section 5.3. These chapters develop the structure on which most of the rest is based. Some parts can be omitted if you need to go to the essentials. These include Sections 2.6 and 4.5. The former relates triangulations with classical topics in polytope theory and the latter is meant as a comprehensive reference list of different ways in which triangulations can be characterized.

Despite our very best intentions there surely remain some errors or typos in the text. We take full responsability for that and plan to post a list of errors and typos at our web sites. Please do let us know via email (our web addresses are listed below) if you find any. (And feel free to write to us with your triangulation stories too!)

This book took too long to write and required the help of many friends and colleagues who taught us and inspired us with their mathematics and wisdom. We are truly grateful to the following people for their ideas, corrections, comments, questions, suggestions, or simply because they patiently kept asking about our seemingly never-ending book project:

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Contents