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Robert P. Gilles

The Cooperative Game Theory of Networks and Hierarchies

Springer

Professor Robert P. Gilles
Management School
Queen's University
25 University Square
Belfast BT7 1NN
United Kingdom
r.gilles@qub.ac.uk

ISSN 0924-6126
ISBN 978-3-642-05281-1 e-ISBN 978-3-642-05282-8
DOI 10.1007/978-3-642-05282-8
Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2010920832

Cover design: WMXDesign GmbH, Heidelberg

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Textbooks on game theory have mainly been rather short on the theory of games in characteristic function form, also known as “cooperative” game theory. In the standard textbooks of Myerson (1991) and Fudenberg and Tirole (1991), the treatment of cooperative game theory has been limited to a single chapter. Only slightly more attention for cooperative games has been offered in the graduate text of Osborne and Rubinstein (1994). The main texts on cooperative game theory are Ichiishi (1983, 1993), Owen (1995), and Peleg and Sudhölter (2007). With the exception of Ichiishi (1993) these texts emphasize the mathematical foundations of the theory rather than its applications to study certain abstract social and economic phenomena. In this text it is my goal to discuss these mathematical foundations of cooperative game theory as well as some applications of these cooperative game theoretic concepts to construct abstract theories of network formation and the functioning of hierarchical authority situations.

Traditionally cooperative game theory has focussed on two fundamental equilibrium notions in the characteristic function form, the *Core* and the *value*. The Core and related equilibrium concepts are founded on the description of power that coalitions of players are potentially able to exercise in the process of allocating collective benefits to the individual players. Different Core-like concepts reflect the potential differences in the power exercised by the different coalitions. These equilibrium concepts are essentially models of how coalitions might affect bargaining or negotiation processes to allocate collectively generated values. Usually these equilibrium concepts identify collections of allocations that are stable against manipulation by various coalitions of players, where such “manipulation” is defined through appropriate rules in the equilibrium concept. The Core is the simplest of these equilibrium notions and is based on the threat of *any* coalition to abandon the negotiation process completely and to divide the value it can generate by itself among its members. This notion is called “blocking”. A Core allocation is now one that cannot be blocked by any coalition of players.

Value theory, on the other hand, refers to an axiomatic equilibrium concept introduced seminally by Shapley (1953), which caused a revolution in the perception of equilibrium analysis. The value is a uniquely constructed allocation rule that satisfies certain desirable properties or “axioms”. This axiomatic approach first describes the desirable properties of an allocation rule and the subsequent analysis shows the uniqueness of the allocation rule founded on these selected axioms. The original Shapley axioms form a mix of properties describing the power of coalitions to influence the allocation process as well as imposing the (relative) fairness of the allocation of resources to individual players.

Hart and Mas-Colell (1989) seminally introduced a further advance in cooperative game theory, by proposing the reduction of each characteristic function form game to a single number. Surprisingly, this important contribution showed that the proposed *potential function* is closely related to Shapley's axiomatic value. This allows an axiomatic characterization of this potential function as well as the Shapley value. In this text I also discuss recent contributions to potential theory that introduce potentials for alternative value functions and related share functions.

I use the fundamental notions of the Core, the Value and the Potential function in the analysis of social communication networks and the exercise of authority in hierarchical organizations. The applications discussed in this text are mainly based on some recent developments in the literature on the cooperative game theoretic analysis of social networks and hierarchical authority structures. I focus this volume mainly on the contributions made by myself and many of my co-authors. Here I refer in particular to my work in collaboration with René van den Brink, Guillermo Owen, Jean Derks, Gerard van der Laan, and the game theory group at Tilburg University, in particular Stef Tijs, Peter Borm, Henk Norde and Marco Slikker.

The cooperative approach to social networks is developed in an extensive literature emanating from the seminal contribution by Myerson (1977). In this text I limit myself to the discussion of directed communication networks rather than the usually assumed undirected communication networks among players. Directed networks also have numerous applications, including the representation of domination situations such as authority structures and sports competitions.

From directed communication situations it is a small step towards hierarchical authority structures. This is the subject of the last chapter in this text. In this chapter I consider the consequences of the exercise of authority in the form of veto power on the productive values that can be genrated by the participating players. In particular, the chapter considers the Shapley-like values that emerge in these authority situations.

Acknowledgements

This text is founded on joint work with several co-authors. In the subsequent chapters I report my joint research with René van den Brink, Guillermo Owen, Jean Derks, Stef Tijs, Henk Norde, and Marco Slikker. In particular I want to point out the cooperation with René van den Brink as the source of numerous ideas and concepts discussed throughout this text. Without his contributions in numerous fields in (applied) cooperative game theory—axiomatic value theory in particular—this text would not have been written. Several concepts discussed in this text are our joint

creations, in particular the β-value for directed networks and the permission values for hierarchical authority situations.

Also, I thank several of my graduate students at Virginia Tech for their feedback on this manuscript. In particular, I thank Zhengzheng Pan for carefully proofreading most of the text. All remaining errors are my own.

Belfast, UK
September 2009

Robert P. Gilles

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