Part IV

Hypercube Embeddings and Designs

Introduction

In Part IV, we study the question of embedding semimetrics isometrically in the hypercube and, in particular, its link with the theory of designs.

Let $t \geq 1$ be an integer. A very simple metric is the *equidistant metric* on n points, denoted by $2t\mathbf{1}_n$, which takes the same value $2t$ on each pair of points. The metric $2t\mathbf{1}_n$ is obviously hypercube embeddable. Indeed, a hypercube embedding of $2t\mathbf{1}_n$ can be obtained by labeling the points by pairwise disjoint sets, each of cardinality t. A basic result established in Chapter 22 is that, for n large enough (e.g., for $n \geq t^2 + t + 3$), this embedding is essentially the unique hypercube embedding of $2t\mathbf{1}_n$. Moreover, for $n \geq t^2$, the existence of another hypercube embedding of $2t\mathbf{1}_n$ depends solely on the existence of a projective plane of order t. In Chapter 23, we further investigate how various hypercube embeddings of $2t\mathbf{1}_n$ arise from designs. We then consider in Chapter 24 some other classes of metrics for which we are able to characterize hypercube embeddability. Typically, these metrics have a small range of values so that one can still take advantage of the knowledge available for their equidistant submetrics. For instance, one can characterize the hypercube embeddable metrics with values in the set $\{a, 2a\}$, or $\{a, b, a+b\}$ (if two of $a, b, a+b$ are odd), where a, b are given integers. Moreover, this characterization yields a polynomial time algorithm for checking hypercube embeddability of such metrics. We recall that, for general semimetrics, it is an NP-hard problem to check whether a given semimetric is hypercube embeddable. Several additional results related to the notion of hypercube embeddability are grouped in Chapter 25, namely, on cut lattices, quasi h-points and Hilbert bases of cuts.

We now recall some definitions and terminology that we use in this part. Let d be a distance on the set $V_n := \{1, \ldots, n\}$. Then, d is said to be *hypercube embeddable* if there exist vectors $u_i \in \{0,1\}^m$ (for some $m \geq 1$) ($i \in V_n$) such that

(a) $$d(i,j) = \| u_i - u_j \|_1$$

for all $i, j \in V_n$. Let M denote the $n \times m$ matrix whose rows are the vectors $u_1, \ldots, u_n$; M is the *realization matrix* of the embedding $u_1, \ldots, u_n$ of d. Any matrix arising as the realization matrix of some hypercube embedding of d is called an *h-realization matrix* of d. Each vector u_i can be seen as the incidence vector of a subset A_i of $\{1, \ldots, m\}$. Hence,

(b) $$d(i,j) = |A_i \triangle A_j|$$

for all $i, j \in V_n$. We also say that the sets $A_1, \ldots, A_n$ form an *h-labeling* of d.

Clearly, if M is an h-realization matrix of d, we can assume without loss of generality that a row of M is the zero vector. This amounts to assuming that one of the points is labeled by the empty set in the corresponding h-labeling of d.

Let $\mathcal{B}$ denote the collection of subsets of V_n whose incidence vectors are the columns of M; $\mathcal{B}$ is a multiset, i.e., it may contain several times the same member. Then, (a) is equivalent to

$$\text{(c)} \qquad d = \sum_{B \in \mathcal{B}} \delta(B).$$

This shows again (recall Proposition 4.2.4) that hypercube embeddable semimetrics are exactly the semimetrics that can be decomposed as a nonnegative integer combination of cut semimetrics. If (c) holds, we also say that $\sum_{B \in \mathcal{B}} \delta(B)$ is a *$\mathbb{Z}_+$-realization* of d. It will be convenient to use both representations (a) (or (b)) and (c) for a hypercube embeddable semimetric d. So, we shall speak of a hypercube embedding (or of an h-labeling of d), and of a $\mathbb{Z}_+$-realization of d. This amounts basically to looking either to the rows, or to the columns of the matrix M.

Recall that d is said to be *h-rigid* if d has a unique $\mathbb{Z}_+$-realization or, equivalently, if d has a unique (up to a certain equivalence) hypercube embedding. Equivalent embeddings were defined in Section 4.3; we remind the definition below. Let $\mathcal{B}$ be a collection of subsets of V_n satisfying (c). If we apply the following operations to $\mathcal{B}$:

(i) delete or add to $\mathcal{B}$ the empty set or the full set V_n,

(ii) replace some $B \in \mathcal{B}$ by its complement $V_n \setminus B$,

then we obtain a new set family $\mathcal{B}'$ satisfying again (c). We then say that $\mathcal{B}$ and $\mathcal{B}'$ are *equivalent* (or define *equivalent* hypercube embeddings), as they yield the same $\mathbb{Z}_+$-realization of d.