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José María Amigó

Permutation Complexity in Dynamical Systems

Ordinal Patterns, Permutation Entropy and All That

Springer

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To my parents

Preface

This is a research book on ordinal patterns, permutation entropy, and complexity, written at graduate level. The common denominator of the different topics presented in its pages is a hypothetical order structure of the state space, substantiated in form of ordinal patterns—permutations defined by the order relations among points in the orbits of dynamical systems. Here the state space is meant to be arbitrary (including discrete sets and n-dimensional intervals), as long as it is totally ordered, and the dynamical systems are meant to include stochastic processes (sometimes called random dynamical systems). Out of the order structure of the state space, a number of constructs will emerge to pave our way as we progress: admissible and forbidden patterns, order isomorphy, metric and topological permutation entropy, discrete entropy, regularity parameters, etc. The relation of these concepts to similar concepts in applied mathematics and computer science will be addressed as well, especially in the introductory part. The final result is a new approach to dynamical complexity characterized by conceptual simplicity, an algebraic flavor, and computational speed. The term *permutation complexity* in the title of this book intends to direct attention to this circle of ideas.

Complexity is a general concept that has different meanings in different contexts. For instance, complexity is related to "incompressibility" in information theory and computer science. In dynamical systems, complexity is usually measured by the topological entropy and reflects roughly speaking, the proliferation of periodic orbits with ever longer periods or the number of orbits that can be distinguished with increasing precision. In physics, the label "complex" is in principle attached to any nonlinear system whose numerical solutions exhibit a chaotic behavior. Neurologists claim that the human brain is the most complex system in the solar system, while entomologists teach us the baffling complexity of some insect societies. The list could be enlarged with examples from geometry, management science, communication and social networks, etc. In this book we will be mainly concerned with complexity from the viewpoint of discrete-time dynamical systems. In particular, permutation complexity refers to the dynamical features captured and quantified by tools based on order relations.

Permutation entropy was introduced in 2002 by C. Bandt and B. Pompe as a measure of complexity in time series. In a nutshell, permutation entropy replaces the probabilities of length-L symbol blocks in the definition of the Shannon entropy by the probabilities of length-L ordinal patterns. Since then this proposal has sparked new lines of research that capitalize on the order structure of the state space. Order is as well at the base of some classical results of combinatorial dynamics (notably, Sarkovskii's theorem), but the focus in these investigations is the periodicity structure of the map. Ordinal patterns provide an akin though different picture: akin because periodic points and ordinal patterns are closely related; different because ordinal patterns are amenable to numerical methods, while periodicity is not. A complete analysis of the relation between ordinal patterns and periodic points is still lacking.

As conventional entropy, permutation entropy comes in metric and topological versions, and these are limits of the corresponding rates of finite order. The metric and topological permutation entropies can be shown to coincide with their conventional counterparts under several assumptions. In applications, permutation entropy rates of finite order may be used to measure the complexity of a finite data sequence. Periodic or quasiperiodic sequences have vanishing or negligible complexity. At the opposite end, independent and identically distributed random sequences (white noise) have asymptotically divergent permutation entropies, owing to the fact that the number of allowed (or "admissible") ordinal patterns grows superexponentially with length. Between both ends lie the kind of sequences we are interested in; their permutation entropy rates of finite order can be calibrated by comparison with the corresponding rates of the white noise.

The study of permutation complexity, which we call *ordinal analysis*, can be envisioned as a new kind of symbolic dynamics whose basic blocks are ordinal patterns. Interesting enough, it turns out that under some mild mathematical assumptions, not all ordinal patterns can be materialized by the orbits of a given one- or multi-dimensional deterministic dynamics, not even if this dynamic is chaotic—contrarily to what happens with the symbol patterns. As a result, the existence of "forbidden" (i.e., not occurring) ordinal patterns is always a persistent dynamical feature, in opposition to properties such as proximity and correlation which die out with time in a chaotic dynamic. Moreover, if an ordinal pattern is forbidden, its absence pervades all longer patterns in form of more missing ordinal patterns, called outgrowth forbidden patterns. Admissible ordinal patterns grow exponentially with length, while forbidden patterns do superexponentially. Since random (unconstrained) dynamics has no forbidden patterns with probability 1, their existence can be used as a fingerprint of deterministic orbit generation.

This book is addressed to both researchers on dynamical systems and complexity and graduate students interested in these subjects. Some topics are already well established; others are asking for generalizations or more comprehensive analyses; still others, like the applications to space–time dynamics, are newcomers. The book consists of ten chapters, plus two technical annexes where the reader can find the mathematical background needed in the main text; overlaps between the main text and the annexes were unavoidable, but they have been kept at a minimum. The topics selected correspond to materials published by the author and collaborators in recent years, although they have been thoroughly revised and eventually reformulated for this occasion. The presentation is a compromise between mathematical rigor and getting the message across in a smooth way. Formal statements of results and their proofs allow knowing exactly which are the assumptions behind them, facilitating at the same time to refer to them from any place in the text. Examples illustrate the theory wherever convenient. Both the main text and the annexes contain also a sufficient number of exercises that invite the reader to explore beyond our exposition. Next we describe briefly the content of the different chapters.

Chapter 1 is an introduction to the main topics of this book, namely, patterns, complexity, and entropy. We show how these concepts are linked—sometimes in unexpected ways—in five different settings: information theory, symbolic dynamics, dynamical systems, computer science, and cellular automata. Ordinal patterns and permutation entropy make their first appearance in the second section, together with the forbidden patterns, one of the main characters of permutation complexity.

Once the stage has been set, Chap. 2 is a brief account on a few applications of ordinal analysis. We review four of them, to wit: entropy estimation, permutation complexity of time series, recovery of control parameters of unimodal maps from symbolic sequences, and characterization of the different kinds of synchronization between chaotic oscillators. This chapter should convey to the reader a first impression of the disparate possibilities of ordinal analysis, before going into technical details in Chaps. 3 through 7.

Chapter 3 is wholly devoted to the study of ordinal patterns and their main properties. Two of them are specially important in applications: existence of forbidden patterns in the orbits of dynamical systems (herein referred only to one-dimensional dynamics) and robustness of admissible and forbidden patterns against observational noise. Forbidden patterns are further classified into two groups: outgrowth and root forbidden patterns. The study of robustness is continued in Chap. 9.

In the relation between maps and the structure of their admissible and forbidden patterns there are far more questions than answers. It is therefore gratifying that this relation can be analyzed with great detail in the case of the shift and signed shift transformations. Due to its length, this topic has been divided into two parts: Chap. 4 and Chap. 5. Signed shifts include the standard ones but their handling is more difficult, and the results gotten till now are not so sharp. By order isomorphy, the results of these two chapters apply to perhaps more interesting cases, like the logistic map, baker map, sawtooth maps.

The next two chapters comprise an in-depth analysis of metric and topological permutation entropies. On defining the metric permutation entropy of maps in Chap. 6, we depart from the original approach to follow basically Kolmogorov's path, based on finite partitions. The pay-off is that the results are not limited to one-dimensional maps. For this reason we have to make a detour over symbolic dynamics (or, equivalently, finite-alphabet information sources), before getting ready to deal with maps. The main outcome is that the metric permutation entropy of ergodic maps coincides with the metric entropy (otherwise called measure-theoretical or Kolmogorov–Sinai entropy) of the map.

The same applies to the topological permutation entropy (Chap. 7), where now expansiveness is called in. An important consequence is the existence of forbidden patterns also in higher dimensional dynamics. Furthermore, numerical simulations provide ample evidence that forbidden patterns is a general feature of deterministic orbit generation.

Discrete entropy (Chap. 8) was proposed (together with the discrete Lyapunov exponent) as a tool of discrete chaos, a generalization of chaos to dynamical systems with discrete state spaces. Our approach follows the work of Bandt and Pompe on permutation entropy of time series. It is proved that discrete entropy converges to its "continuous" counterpart in an adequate sense.

Having shown in Chap. 7 that the existence of forbidden patterns is a landmark of determinism, Chap. 9 grapples with the implementation of this fact, the main obstacle being that real data are finite and noisy. The properties of ordinal patterns studied in Chap. 3 come here to the rescue, as well as the "dynamical robustness" discussed in the first section. Two methods are proposed, based on (i) the number of missing ordinal patterns and (ii) the distribution of visible ordinal patterns. The second resorts to a chi-square test, the null hypothesis being that the time series is white noise; its performance compares favorably to some widely used tests of statistical independence.

Cellular automata and coupled map lattices are, so to speak, toy models for real physics. And yet, what these dynamical systems lack in sophistication as compared to the usual space–time systems, they more than make up for in conceptual simplicity and modelization power. On applying some tools of ordinal analysis to cellular automata and coupled map lattices, as done in Chap. 10, we put to test the capabilities of this approach to discern different temporal structures in spatially extended systems. The task is formidable: trying to reduce the behavior of a space–time system to just a parameter seems to be more than what one could reasonably ask for. Nevertheless, the results reported in Chap. 10 are encouraging.

The book concludes with Chap. 11, where we remind the main messages of ordinal analysis and permutation complexity, gather some open problems scattered in the preceding chapters, and suggest future lines of research.

Much labor will be necessary to survey the full potential of ordinal analysis and the intricacies of permutation complexity at theoretical and practical levels. This book should be considered as a contribution to this task. One of the main challenges of complexity theory is to design conceptual and numerical tools to study, classify, and quantify the different degrees of complexity found in our mathematical models of the world around. Think, for example, of turbulence in fluid mechanics or the asymptotic behavior of cellular automata and coupled map lattices. Nonlinear physics has developed a battery of instruments that go by the name of power spectra, Lyapunov exponents, fractal dimensions of attractors, order parameters, etc. On the mathematical side, ergodic theory and topological dynamics study general properties of systems evolving in time. These disciplines have provided plenty of handles to understand complex dynamics, like deep concepts, invariants for classification purposes (most notably, the entropy), prototypes, and powerful theoretical and practical techniques. But order relations have been less exploited. One possible reason is that order relations are not invariant under metric and topological isomorphisms, which consistently only address measure-theoretical and topological properties. We hope that this book on permutation complexity convincingly shows that properties related to the temporal (and eventually also spatial) structure of a dynamics are useful and worth researching.

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Elche, Spain José María Amigó

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